Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 3, No. 1, 2025, Pages 23-33
DOI: 10.47831/mjpas.v3i1.112

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage:
https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

Reliability Estimation for Inverted Kumaraswamy Distribution:A Simulation Study

Zainab Saddi Noori[1]*, Makki A. Mohammed Salih[2]

[1]Department of Mathematics, Open Educational College-Baghdad-Iraq
[2]Department of Mathematics, College of Education, Mustansiriyah University-Baghdad-Iraq

* Corresponding author E-mail: zainabsaadi97@gmail.com

Article Info.

Article history:

Received
25 January 2024

Accepted
25 April 2024

Publishing
30 January 2025

Abstract

In this paper, the parameters and reliability of the inverse Kumaraswamy distribution were estimated through estimation methods namely: Moments, Modified Moments, Percentiles, White, Modified White and Maximum Likelihood. The simulation was used for estimation through samples of different sizes and then the mean square error criterion was used to compare between them. The results obtained from parameter estimators are shown: As for the shape parameter, the Percentile method is the best in most of the experiments that were used, for the exponentiated parameter, the Maximum Likelihood method is the best in most of the experiments conducted. As for reliability, its preference varies between the Maximum Likelihood method and the Modified White method.

The official journal published by the College of Education at Mustansiriya University

Keywords: Inverted Kumaraswamy Distribution, Estimation Methods, Simulation and Mean Square Error.

1. Introduction

The Kumaraswamy distribution (Kum) was proposed by Kumaraswamy (1980) in which the formulas of probability density function PDF and the cumulative distribution function CDF are as follows: [1]

$$f(y,\alpha,\sigma)=\alpha\sigma y^{\alpha-1}(1-y^{\alpha})^{\sigma-1} \quad ,0\le y\le 1 \ , \alpha,\sigma>0$$

$$F(y,\alpha,\sigma)=1-(1-y^{\alpha})^{\sigma}$$

Several researches have studied it, like Javanshiri et al. (2015), they proposed and studied Exp-Kumaraswamy distribution. They derived some of its properties like the density function, hazared rate function, quantile function, moments, skewness and kurtosis [2]. In (2016), Khan et al. they proposed a generalization of the Kumaraswamy distribution referred to as the transmuted Kumaraswamy distribution. The new transmuted distribution is developed using the quadratic rank transmutation map[3]. In (2017), both Makki and Al-Altawil studied the estimation of the parameters and reliability of the two-parameter Kumaraswamy distribution. They proposed a new distribution for the (Kum) by adding a third parameter, which is the location parameter and they presented a study in estimating the new distribution for its parameters and reliability[4]. In (2020), Mohamoud and Refaey they introduced the generalized inverted Kumaraswamy distribution and presented some properties of the distribution and they used the Maximum Likelihood and Bayesian methods for estimating the model parameters[5]. In (2021), Kadir et al. they suggested a new statistical distribution called DUS-Kumaraswamy[6]. In (2023), Ishaq at al. they presented a new distribution called Log-Kumaraswamy distribution. They estimated its parameters by using the maximum likelihood, weighted least squares, least squares and maximum products of spacing methods [7]. In (2024), Mohmoud and Saad they used the principle of maximum entropy to propose new estimators for the Kumaraswamy parameters and compared with Maximum Likelihood and Bayesian estimation methods [8].

The inverse Kumaraswamy (IKum) distribution was proposed by Al-Fattah et al. (2017), where they derived the functions and properties of this distribution. The most important functions for this distribution are as follows: [9]

$$f(x;\alpha,\sigma)=\alpha\sigma(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\sigma-1}, x>0 \tag{1}$$

$$F(x;\alpha,\sigma)=(1-(1+x)^{-\alpha})^{\sigma}, x>0 \tag{2}$$

$$R(t;\alpha,\sigma)=1-(1-(1+t)^{-\alpha})^{\sigma}, t>0 \qquad (3)$$

Whereas α and σ are shape and exponentiated parameters $\alpha,\sigma>0$

From equation (2), the quintile of the (IKum) distribution is:

$$x_F=\left(1-(F)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}-1 \qquad (4)$$

In this research, we will study the estimation of the reliability of the Inverted Kumaraswamy Distribution through the estimation methods in the second section, which were not used for this distribution, according to the simulation data that was taken in the third section. In the fourth section, the results were extracted and placed in tables for the purpose of comparison.

2. Estimation Methods

In this section, the derivation of estimators and estimation methods for the (IKum) distribution will be presented

2.1 Estimating the initial values of the parameters

To estimate the initial values of the parameters, the median of the sample is extracted after generating it through simulation and then the value of this median is equated with the formula of the median of the distribution. Through this relationship, two formulas are extracted for the initial values of the two parameters[10], as follows:

The median for (IKum) distribution is: [9]

$$x_{med}=\left(1-(0.5)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}-1 \qquad (5)$$

From equation (5)

$$x_{med}+1=\left(1-(0.5)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}} \qquad (6)$$

By taking the natural logarithm for equation (6), getting:

$$\alpha ln(x_{med}+1)=-ln\left(1-(0.5)^{\frac{1}{\sigma}}\right)$$

$$\hat{\alpha}_0=\frac{-ln\left(1-(0.5)^{\frac{1}{\sigma}}\right)}{ln(x_{med}+1)} \qquad (7)$$

Now, from equation (5)

$$x_{med}+1=\frac{1}{\left(1-(0.5)^{\frac{1}{\sigma}}\right)^{\frac{1}{\alpha}}}$$

$$1-(0.5)^{\frac{1}{\sigma}}=\frac{1}{(x_{med}+1)^{\alpha}}$$

$$\frac{1}{\sigma}ln(0.5)=ln\left(1-\frac{1}{(x_{med}+1)^{\alpha}}\right)$$

$$\hat{\sigma}_0=\frac{ln(0.5)}{ln\left(1-\frac{1}{(x_{med}+1)^{\alpha}}\right)} \qquad (8)$$

Both equations (7) and (8) represent the formulas for the initial parameters and x_{med} is the median of sample.

2.2 Moments Method (MOM)

This method is one of the common methods used in many studies and research, which depends on the sampling moment (m_k), the population moment (M_k) and their equality with each other and then formulas for distribution estimates are extracted.[11]

$$M_k = m_{k,}\ k = 1,2,3,\dots,r \tag{9}$$

The k^{th} moments a bout zero is: [9]

$$M_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sigma B\left(1-\frac{j}{\alpha},\sigma\right), k = 1,2,3,\cdots,r\ ,\alpha > j, j = 0,1,2,3,\cdots,k \tag{10}$$

Whereas $B(.,.)$ is the beta function

When $(k = 1)$ in Equations (9) and (10), we obtain the first estimator as follows:

$M_1 = m_1$

Since,

$M_1 = E(x) = \frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)} - 1$ and $m_1 = E(x) = \frac{\sum_{j=1}^{m} x_j}{m}$ then

$$\frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)} - 1 = \frac{\sum_{j=1}^{m} x_j}{m}$$

$$\frac{\Gamma(1+\sigma)\left(\frac{-1}{\alpha}\right)\Gamma\left(\frac{-1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)} = \frac{\sum_{j=1}^{m} x_j}{m} + 1$$

$$\hat{\alpha}_{MOM} = -\frac{\Gamma(1+\sigma_0)\Gamma\left(\frac{-1}{\alpha_0}\right)}{\left(\frac{\sum_{j=1}^{m} x_j}{m}+1\right)\Gamma\left(1-\frac{1}{\alpha_0}+\sigma_0\right)} \tag{11}$$

Now, by taking $(k = 2)$ in equations (9) and (10), getting:

$M_2 = m_2$

Since, $M_2 = E(x^2) = 1 - 2\frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)} + \frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{2}{\alpha}+\sigma\right)}$ and $m_2 = E(x^2) = \frac{\sum_{j=1}^{m} x_j^2}{m}$ then

$$1 - 2\frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)} + \frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{2}{\alpha}+\sigma\right)} = \frac{\sum_{j=1}^{m} x_j^2}{m}$$

$$\frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{2}{\alpha}+\sigma\right)} = \frac{\sum_{j=1}^{m} x_j^2}{m} - 1 + 2\frac{\sigma\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)}$$

$$\hat{\sigma}_{MOM} = \frac{\frac{\sum_{j=1}^{m} x_j^2}{m} - 1 + 2\frac{\sigma_0\Gamma(\sigma_0)\Gamma\left(1-\frac{1}{\alpha_0}\right)}{\Gamma\left(1-\frac{1}{\alpha_0}+\sigma_0\right)}}{\frac{\Gamma(\sigma_0)\Gamma\left(1-\frac{2}{\alpha_0}\right)}{\Gamma\left(1-\frac{2}{\alpha_0}+\sigma_0\right)}} \tag{12}$$

Whereas, $\alpha_0 > 2$.

Substitute equations (11) and (12) in equation (3), getting:

$$\hat{R}_{MOM}(t) = 1 - \left(1 - (1+t)^{-\hat{\alpha}_{MOM}}\right)^{\hat{\sigma}_{MOM}} \tag{13}$$

2.3 Modified Moment's Method (MD)

One of the characteristics of this method is to use the population expectation at a certain order statistic and equate it with the distribution function at this same order statistic[12]. The sample variance is also used with the population variance by equating them and extracting the formulas for distribution estimates, as follow[13]

$$F\left(x_{(m)}\right) = E\left(\hat{F}\left(x_{(m)}\right)\right) \tag{14}$$

$\hat{F}\left(x_{(m)}\right)$ is unbiased value for distribution function $F\left(x_{(m)}\right)$ and replacing $E\left(\hat{F}\left(x_{(m)}\right)\right)$ by

$$p_j = \frac{j}{m+1}, \qquad j = 1,2,\dots,m \tag{15}$$

Getting:

$$E\left(\hat{F}\left(x_{(m)}\right)\right) = \frac{m}{m+1} \tag{16}$$

By substituting $\left(x_{(m)}\right)$ into equation (2) leads to:

$$F\left(x_{(m)}\right) = \left(1-\left(1+x_{(m)}\right)^{-\alpha}\right)^{\sigma} \tag{17}$$

Using equations (16) and (17) in equation (14) leads to:

$$\left(1+x_{(m)}\right)^{-\alpha} = 1-\left(\frac{m}{m+1}\right)^{\frac{1}{\sigma}} \tag{18}$$

$$-\alpha ln\left(1+x_{(m)}\right) = ln\left(1-\left(\frac{m}{m+1}\right)^{\frac{1}{\sigma}}\right)$$

$$\hat{\alpha}_{MD} = -ln\left(1-\left(\frac{m}{m+1}\right)^{\frac{1}{\sigma_0}}\right)/ln\left(1+x_{(m)}\right) \tag{19}$$

Now, to find the second estimator, we use the property of equalizing the variance of the distribution with the variance of the sample, as follows:

$$\sigma\frac{\Gamma(\sigma)\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{2}{\alpha}+\sigma\right)} - \left(\sigma\frac{\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)}\right)^2 = \frac{\sum_{j=1}^{m}\left(x_j-\bar{x}\right)^2}{m-1}$$

$$\sigma^2\left(\frac{\Gamma(\sigma)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\sigma\right)}\right)^2 = \sigma\frac{\Gamma(\sigma)\Gamma\left(1-\frac{2}{\alpha}\right)}{\Gamma\left(1-\frac{2}{\alpha}+\sigma\right)} - \frac{\sum_{j=1}^{m}\left(x_j-\bar{x}\right)^2}{m-1}$$

$$\hat{\sigma}_{MD} = \sqrt{\frac{\sigma_0\frac{\Gamma(\sigma_0)\Gamma\left(1-\frac{2}{\alpha_0}\right)}{\Gamma\left(1-\frac{2}{\alpha_0}+\sigma_0\right)} - \frac{\sum_{j=1}^{m}\left(x_j-\bar{x}\right)^2}{m-1}}{\left(\frac{\Gamma(\sigma_0)\Gamma\left(1-\frac{1}{\alpha_0}\right)}{\Gamma\left(1-\frac{1}{\alpha_0}+\sigma_0\right)}\right)^2}} \tag{20}$$

Whereas, $\bar{x} = \frac{\sum_{j=1}^{m}x_j}{m}$ and $\alpha_0 > 2$.

Substitute equations (19) and (20) in equation (3), getting:

$$\hat{R}_{MD}(t) = 1-\left(1-(1+t)^{-\hat{\alpha}_{MD}}\right)^{\hat{\sigma}_{MD}} \tag{21}$$

2.4 Percentile Method (PE)

The quintile equation (4), is used to estimate the distribution parameters by zeroing and squaring them and then taking the partial derivative of the formula with respect of the two parameters, as follows:[14] [15]

$$s = \sum_{j=1}^{m}\left(\left(x_{(j)}+1\right)-\left(1-(F)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}\right)^{2} = 0 \tag{22}$$

Replacing F, equation (22), by equation (15) leads to:

$$s = \sum_{j=1}^{m}\left(\left(x_{(j)}+1\right)-\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}\right)^{2} = 0 \tag{23}$$

Now the partial derivatives of the equation (23) are taken, as follows:

$$\frac{\partial s}{\partial \alpha} = 2\sum_{j=1}^{m}\left(\left(x_{(j)}+1\right)-\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}\right)\left(-\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)\frac{1}{\alpha^2}\right) = 0$$

$$\sum_{j=1}^{m}\left(x_{(j)}+1\right)\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right) = \sum_{j=1}^{m}\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{2}{\alpha}} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right) \tag{24}$$

Taking the natural logarithm for equation (24), getting:

$$\sum_{j=1}^{m} ln\left(x_{(j)}+1\right)-\frac{1}{\alpha}\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)+\sum_{j=1}^{n} lnln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)$$

$$= -\frac{2}{\alpha}\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)+\sum_{j=1}^{m} lnln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)$$

$$-\frac{2}{\alpha}\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)+\frac{1}{\alpha}\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right) = \sum_{j=1}^{m} ln\left(x_{(j)}+1\right)$$

$$\frac{-1}{\alpha}\left(\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)\right) = \sum_{j=1}^{m} ln\left(x_{(j)}+1\right)$$

$$\hat{\alpha}_{PE} = \frac{-\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma_0}}\right)}{\sum_{j=1}^{m} ln\left(x_{(j)}+1\right)} \tag{25}$$

Now,

$$\frac{\partial s}{\partial \sigma} = 2\sum_{j=1}^{m}\left(\left(x_{(j)}+1\right)-\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}}\right)\left(-\left(-\frac{1}{\alpha}\right)\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}-1}\right)\left(-\left(p_j\right)^{\frac{1}{\sigma}} ln\left(p_j\right)\left(-\frac{1}{\sigma^2}\right)\right) = 0$$

$$\sum_{j=1}^{m}\left(x_{(j)}+1\right)\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}-1}\left(p_j\right)^{\frac{1}{\sigma}} ln\left(p_j\right) = \sum_{j=1}^{m}\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{2}{\alpha}-1}\left(p_j\right)^{\frac{1}{\sigma}} ln\left(p_j\right) \tag{26}$$

The natural logarithm for equation (26) is

$$\sum_{j=1}^{m} ln\left(x_{(j)}+1\right)+\sum_{j=1}^{m} ln\left(1-\left(p_j\right)^{\frac{1}{\sigma}}\right)^{-\frac{1}{\alpha}-1}+\frac{1}{\sigma}\sum_{j=1}^{m} ln\left(p_j\right)+\sum_{j=1}^{m} lnln\left(p_j\right) =$$

$$\sum_{j=1}^{m} ln\left(1-(p_j)^{\frac{1}{\sigma}}\right)^{-\frac{2}{\alpha}-1}+\sum_{j=1}^{m} ln(p_j)^{\frac{1}{\sigma}}+\sum_{j=1}^{m} lnln(p_j)$$

$$\hat{\sigma}_{PE}=\frac{\sum_{j=1}^{m} ln(p_j)}{\sum_{j=1}^{m} ln\left(1-(p_j)^{\frac{1}{\sigma_0}}\right)^{-\frac{2}{\alpha_0}-1}+\sum_{j=1}^{m} ln(p_j)^{\frac{1}{\sigma_0}}-\sum_{j=1}^{m} ln(x_{(j)}+1)-\sum_{j=1}^{m} ln\left(1-(p_j)^{\frac{1}{\sigma_0}}\right)^{-\frac{1}{\alpha_0}-1}} \tag{27}$$

Substitute equations (25) and (27) in equation (3), getting:

$$\hat{R}_{PE}(t)=1-\left(1-(1+t)^{-\hat{\alpha}_{PE}}\right)^{\hat{\sigma}_{PE}} \tag{28}$$

2.5 White Method (W)

Reliability is used to estimate the two distribution parameters by converting its mathematical formula into a linear regression equation formula and then extracting the formula for the estimators of the two parameters, as follows:[16] [17]

From equation (3)

$$1-R(x)=(1-(1+x)^{-\alpha})^{\sigma}$$

Equating equation (2) with equation (15), getting:

$$F(x_j)=\frac{j}{m+1}\ ,j=1,2,\dots,m$$

Since $1-R(x_j)=F(x_j)=\frac{j}{m+1}$ then:

$$\frac{j}{m+1}=\left(1-\left(1+x_{(j)}\right)^{-\alpha}\right)^{\sigma}$$

$$1-\frac{j^{\frac{1}{\sigma}}}{(m+1)^{\frac{1}{\sigma}}}=\left(1+x_{(j)}\right)^{-\alpha}$$

$$(m+1)^{\frac{1}{\sigma}}-j^{\frac{1}{\sigma}}=(m+1)^{\frac{1}{\sigma}}\left(1+x_{(j)}\right)^{-\alpha} \tag{29}$$

Taking the natural logarithm for equation (29), getting:

$$ln\left((m+1)^{\frac{1}{\sigma}}-j^{\frac{1}{\sigma}}\right)=\frac{1}{\sigma}ln(m+1)+\alpha\left(-ln\left(1+x_{(j)}\right)\right) \tag{30}$$

The formula for linear regression is:

$$\delta_j=\varphi+\omega\emptyset_j+\epsilon_j \tag{31}$$

By comparing the two equations (30) and (31), getting:

$$\delta_j=ln\left((m+1)^{\frac{1}{\sigma}}-j^{\frac{1}{\sigma}}\right),\bar{\delta}=\frac{\sum_{j=1}^{m}\delta_j}{m}\quad ,j=1,2,\dots,m \tag{32}$$

$$\varphi=\frac{\ln(m+1)}{\sigma} \tag{33}$$

$$\omega=\alpha \tag{34}$$

$$\emptyset_j=-ln\left(1+x_{(j)}\right)1,2,\dots,m \tag{35}$$

$$\overline{\emptyset}=\frac{\sum_{j=1}^{m}\widehat{\emptyset}_j}{n}\quad ,j=1,2,\dots,m \tag{36}$$

The following formula is to find the approximate value of (ω)

$$\widehat{\omega}_W=\frac{\sum_{j=1}^{m}\left(\widehat{\emptyset}_j-\overline{\emptyset}\right)\left(\delta_j-\bar{\delta}\right)}{\sum_{j=1}^{m}\left(\widehat{\emptyset}_j-\overline{\emptyset}\right)^2} \tag{37}$$

From the two equations (34) and (37) it is obtained:

$$\hat{\alpha}_W=\widehat{\omega}_W \tag{38}$$

From equation (31), getting:

$$\hat{\varphi}_W = \bar{\delta} - \widehat{\omega}_W \overline{\emptyset} \tag{39}$$

Substitute equation (39) in equation (33), getting:

$$\frac{\ln(m+1)}{\hat{\sigma}_w} = \bar{\delta} - \widehat{\omega}_W \overline{\emptyset}$$

$$\hat{\sigma}_W = \frac{ln(m+1)}{\bar{\delta} - \hat{\alpha}_W \overline{\emptyset}} \tag{40}$$

Substitute equations (38) and (40) in equation (3), getting:

$$\hat{R}_W(t) = 1 - \left(1 - (1+t)^{-\hat{\alpha}_W}\right)^{\hat{\sigma}_w} \tag{41}$$

2.6 Modified White Method (MW)

This method differs from White's method by converting the risk function formula to a linear regression formula to obtain the formulas for the estimators of the two distribution parameters.[18]
From hazard function for (IKum) distribution is: [9]

$$\frac{1}{h(x)} = \frac{1 - (1-(1+x)^{-\alpha})^{\sigma}}{\alpha\sigma(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\sigma-1}}$$

$$\frac{1}{h(x)} = \frac{1}{\alpha\sigma(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\sigma-1}} - \frac{1}{\frac{\alpha\sigma(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}}$$

$$\frac{1}{\frac{1}{h(x)} + \frac{1-(1+x)^{-\alpha}}{\alpha\sigma(1+x)^{-(\alpha+1)}}} = \alpha\sigma(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\sigma-1} \tag{42}$$

$$ln\left(\frac{1}{\frac{1}{h(x)} + \frac{1-(1+x)^{-\alpha}}{\alpha\sigma(1+x)^{-(\alpha+1)}}}\right) - ln(\sigma) + (\alpha+1)ln(1+x) = ln(\alpha) + (\sigma-1)ln(1-(1+x)^{-\alpha}) \tag{43}$$

By comparison equation (43) with equation (31), getting:

$$\delta_j = ln\left(\frac{1}{\frac{1}{h_0(x_{(j)})} + \frac{1-\left(1+x_{(j)}\right)^{-\alpha_0}}{\alpha_0\sigma_0\left(1+x_{(j)}\right)^{-(\alpha_0+1)}}}\right) - ln(\sigma_0) + (\alpha_0+1)ln\left(1+x_{(j)}\right), j = 1,2 \dots, m \tag{44}$$

Whereas; $h_0\left(x_{(j)}\right) = \frac{\alpha_0\sigma_0\left(1+x_{(j)}\right)^{-(\alpha_0+1)}\left(1-\left(1+x_{(j)}\right)^{-\alpha_0}\right)^{\sigma_0-1}}{1-\left(1-\left(1+x_{(j)}\right)^{-\alpha_0}\right)^{\sigma_0}}$ and $\bar{\delta} = \frac{\sum_{j=1}^{m}\delta_j}{m}$

$$\varphi = ln(\alpha) \Rightarrow \alpha = exp(\varphi) \tag{45}$$

$$\omega = \sigma - 1 \Rightarrow \sigma = \omega + 1 \tag{46}$$

$$\emptyset_j = ln\left(1 - \left(1+x_{(j)}\right)^{-\alpha}\right) \tag{47}$$

$$\widehat{\emptyset}_j = ln\left(1 - \left(1+x_{(j)}\right)^{-\alpha_0}\right), \overline{\emptyset} = \frac{\sum_{j=1}^{m}\widehat{\emptyset}_j}{m}, j = 1,2,\dots,m \tag{48}$$

Substituting equations (44) and (48) into equation (37), getting:

$$\hat{\sigma}_{MW} = \widehat{\omega}_{MW} + 1 \tag{49}$$

From equation (31), getting:

$$\hat{\varphi}_{MW} = \bar{\delta} - \hat{\sigma}_{MW}\overline{\emptyset} \tag{50}$$

Substituting equation (50) into equation (45), getting:

$$\hat{\alpha}_{MW} = exp\left(\bar{\delta} - \hat{\sigma}_{MW}\overline{\emptyset}\right) \tag{51}$$

Substitute equations (49) and (51) in equation (3), getting:

$$\hat{R}_{MW}(t) = 1 - \left(1 - (1+t)^{-\hat{\alpha}_{MW}}\right)^{\hat{\sigma}_{MW}} \quad (52)$$

2.7 Maximum Likelihood Method (ML)

This method is one of the famous methods that competes with most other estimation methods in its accuracy. The method of this method depends on multiplying the scientific density function of the distribution and then taking the natural logarithm [19]as follows[20]
The likelihood function PDF for (IKum) distribution is:[9]

$$\mathcal{L}(\alpha,\sigma;x_1,x_2,\dots,x_m) = \alpha^m\sigma^m\left(\prod_{j=1}^{m}\left(1+x_j\right)^{-(\alpha+1)}\right)\left(\prod_{j=1}^{m}\left(1-\left(1+x_j\right)^{-\alpha}\right)^{\sigma-1}\right) \quad (53)$$

The natural logarithm for equation (53), is

$$\mathcal{LL}(\alpha,\sigma;x_1,x_2,\dots,x_m) = mln(\alpha) + mln(\sigma) - (\alpha+1)\sum_{j=1}^{m} ln\left(1+x_j\right) + (\sigma-1)\sum_{j=1}^{m} ln\left(1-\left(1+x_j\right)^{-\alpha}\right) \quad (54)$$

The partial derivative for equation (54) with respect to (α) and equal to zero.

$$\frac{\partial\mathcal{LL}}{\partial\alpha} = \frac{m}{\alpha} - \sum_{j=1}^{m} ln\left(1+x_j\right) + (\sigma-1)\sum_{j=1}^{m}\frac{\left(1+x_j\right)^{-\alpha} ln\left(1+x_j\right)}{1-\left(1+x_j\right)^{-\alpha}} = 0$$

$$\hat{\alpha}_{ML} = \frac{m}{\sum_{j=1}^{m} ln\left(1+x_j\right) - (\sigma_0-1)\sum_{j=1}^{m}\frac{\left(1+x_j\right)^{-\alpha_0} ln\left(1+x_j\right)}{1-\left(1+x_j\right)^{-\alpha_0}}} \quad (55)$$

The partial derivative for equation (54) with respect to (σ) and equal to zero.

$$\frac{\partial\mathcal{LL}}{\partial\sigma} = \frac{m}{\sigma} + \sum_{j=1}^{m} ln\left(1-\left(1+x_j\right)^{-\alpha}\right) = 0$$

$$\hat{\sigma}_{ML} = \frac{-m}{\sum_{j=1}^{m} ln\left(1-\left(1+x_j\right)^{-\alpha_0}\right)} \quad (56)$$

Substitute equations (55) and (56) in equation (3), getting:

$$\hat{R}_{ML}(t) = 1 - \left(1 - (1+t)^{-\hat{\alpha}_{ML}}\right)^{\hat{\sigma}_{ML}} \quad (57)$$

3. Simulation

The simulation method was used to estimate the parameters, and the estimation steps were as follows:

1. Choosing sample size $(n = 15,40,70,100)$ and replicated sample (N=1000).
2. The following (Table 1) shows the default values of the two parameters and the number of experiments taken.

Table 1. The default values of the parameters

Exp.→ / Par.↓	E_1	E_2	E_3	E_4
α	3	1	1.5	2.5
σ	0.5	1	3	4

3. Using the mean squares error MSE criterion:
   $MSE(\hat{\theta}) = \frac{\sum_{i=1}^{n}(\hat{\theta}_i-\theta)^2}{N}$, where $(\hat{\theta})$ is estimator for parameter (θ).
4. Using MATLAB language version R2015b to find all results.

4. Results and Conclusion

Table 2. MSE values using (E_1)

Est.	n	MOM	MD	PE	W	MW	ML	Best
$\hat{\alpha}$	15	0.17675	1.72740	0.58844	1.40111	0.50105	0.24632	MOM
$\hat{\sigma}$		0.04887	1.19055	0.00824	0.00594	0.02491	0.00981	W
$\hat{R}$		0.00954	0.00828	0.00084	0.00020	0.00117	0.00007	ML
$\hat{\alpha}$	40	0.13480	0.86208	0.24640	0.61171	0.37044	0.12789	ML
$\hat{\sigma}$		0.04137	0.76821	0.00731	0.00481	0.01053	0.00347	ML
$\hat{R}$		0.00502	0.00386	0.00030	0.00095	0.00017	0.00002	ML
$\hat{\alpha}$	70	0.12770	0.60629	0.13153	0.37086	0.31664	0.08756	ML
$\hat{\sigma}$		0.02977	0.21684	0.00598	0.00427	0.00690	0.00184	ML
$\hat{R}$		0.00437	0.00314	0.00020	0.00072	0.00016	0.000001	ML
$\hat{\alpha}$	100	0.10782	0.51368	0.10330	0.29780	0.24038	0.06959	ML
$\hat{\sigma}$		0.02948	0.07657	0.00540	0.00332	0.00506	0.00140	ML
$\hat{R}$		0.00284	0.00236	0.00012	0.00047	0.00012	0.0000005	ML

Table 3. MSE values using (E_2)

Est.	n	MOM	MD	PE	W	MW	ML	Best
$\hat{\alpha}$	15	----------	----------	0.03005	0.09926	0.10251	0.44097	PE
$\hat{\sigma}$		----------	----------	1.74322	0.20424	0.10273	0.04177	ML
$\hat{R}$		----------	----------	0.00092	0.00494	0.00154	0.00087	ML
$\hat{\alpha}$	40	----------	----------	0.01470	0.04630	0.05302	0.09703	PE
$\hat{\sigma}$		----------	----------	1.30616	0.07191	0.04753	0.01382	ML
$\hat{R}$		----------	----------	0.00071	0.00165	0.00041	0.00023	ML
$\hat{\alpha}$	70	----------	----------	0.00786	0.02687	0.03111	0.05910	PE
$\hat{\sigma}$		----------	----------	0.84838	0.03975	0.02747	0.00833	ML
$\hat{R}$		----------	----------	0.00060	0.00069	0.00020	0.00006	ML
$\hat{\alpha}$	100	----------	----------	0.00592	0.01947	0.02177	0.01136	PE
$\hat{\sigma}$		----------	----------	0.54618	0.02415	0.02025	0.00600	ML
$\hat{R}$		----------	----------	0.00042	0.00035	0.00009	0.00003	ML

Table 4. MSE values using (E_3)

Est.	n	MOM	MD	PE	W	MW	ML	Best
$\hat{\alpha}$	15	----------	----------	0.02660	0.14608	0.08225	0.91793	PE
$\hat{\sigma}$		----------	----------	2.80879	3.18208	0.88361	0.37896	ML
$\hat{R}$		----------	----------	0.00371	0.00650	0.00068	0.00091	MW
$\hat{\alpha}$	40	----------	----------	0.01115	0.07283	0.05192	0.41576	PE
$\hat{\sigma}$		----------	----------	2.64116	1.55998	0.43422	0.12831	ML
$\hat{R}$		----------	----------	0.00115	0.00405	0.00017	0.00073	MW
$\hat{\alpha}$	70	----------	----------	0.00648	0.04549	0.03442	0.25422	PE
$\hat{\sigma}$		----------	----------	2.55145	0.90062	0.24902	0.07461	ML
$\hat{R}$		----------	----------	0.00049	0.00301	0.00007	0.00052	MW
$\hat{\alpha}$	100	----------	----------	0.00471	0.03147	0.02677	0.17824	PE
$\hat{\sigma}$		----------	----------	2.51201	0.54440	0.16512	0.05734	ML
$\hat{R}$		----------	----------	0.00028	0.00196	0.00003	0.00031	MW

Table 5. MSE values using (E_4)

Est.	n	MOM	MD	PE	W	MW	ML	Best
$\hat{\alpha}$	15	0.12036	0.29600	0.05644	0.29994	0.19729	1.65564	PE
$\hat{\sigma}$		3.06965	3.55327	2.96482	4.23708	1.56224	0.59137	ML
$\hat{R}$		0.02147	0.00423	0.00179	0.00405	0.00031	0.00115	MW
$\hat{\alpha}$	40	0.06238	0.20559	0.02203	0.16160	0.11242	0.81086	PE
$\hat{\sigma}$		2.77026	2.99901	2.89955	2.97571	0.74460	0.24547	ML
$\hat{R}$		0.01416	0.00363	0.00026	0.00399	0.00012	0.00045	MW
$\hat{\alpha}$	70	0.04099	0.16525	0.01406	0.11728	0.08457	0.59462	PE
$\hat{\sigma}$		2.57936	2.99440	2.81223	2.23409	0.44965	0.14733	ML
$\hat{R}$		0.00788	0.00361	0.00015	0.00369	0.00004	0.00024	MW
$\hat{\alpha}$	100	0.03065	0.14492	0.01014	0.08279	0.06272	0.51005	PE
$\hat{\sigma}$		2.39113	2.74478	2.66965	1.45190	0.31450	0.10145	ML
$\hat{R}$		0.00766	0.00349	0.00008	0.00288	0.00003	0.00022	MW

From the results in the table above, the following is evident:

- From the results of the first experiment (E_1), when $(\alpha > \sigma)$, it is clear that the (ML) is the best in most of the sample size, except in the sample size (m=15), we notice that the (MOM) is the best in estimating (α), while in estimating (σ), the (W) is the best in the same sample size.
- The results of the second experiment (E_2), when $(\alpha = \sigma)$, show that (PE) is the best in estimating the parameter (α) for all four sample sizes, for the (σ) parameter and reliability (ML) is the best for all sample sizes.
- In the third and fourth experiments (E_3, E_4), when $(\alpha < \sigma)$, we notice that there is a difference in the preference of estimation methods, which is (PE) is the best in estimating the (α) parameter, while the (ML) is the best in estimating the (σ) parameter and (MW) is the best in estimating reliability for all cases of sample sizes.
- One of which is the lack of results for the two methods of (MOM) and (MD), because there is a condition in the general formula for the moments for this distribution that the value of the parameter (α) must be greater than 2.
- The results appeared, in general, that the shape parameter, the Percentile method is the best in most of the experiments that were used, the Maximum Likelihood method is the most effective for the exponentiated parameter. As for reliability, there are difference in preference between the Maximum Likelihood method and the Modified White method.

Acknowledgement

We would like to thank the College of Education/ MUOSTANSIRIYAH UNIVERSITY, represented by its magazine, Mustansiriyah Journal of Pure and Applied Sciences (https://mjpas.uomustansiriyah.edu.iq) , for their efforts to publish this research.

Reference

[1] P. Kumaraswamy, "A generalized probability density function for double-bounded random processes," *J. Hydrol.*, vol. 46, no. 1–2, pp. 79–88, 1980.

[2] Z. Javanshiri, A. Habibi Rad, and N. R. Arghami, "Exp-Kumaraswamy distributions: Some properties and applications," *J. Sci. Islam. Repub. Iran*, vol. 26, no. 1, pp. 57–69, 2015.

[3] M. S. Khan, R. King, and I. L. Hudson, "Transmuted kumaraswamy distribution," *Stat. Transit. new Ser.*, vol. 2, no. 17, pp. 183–210, 2016.

[4] "Altawil, J. A. & Mohammed Salih, M. A. (2017). New parameterization for Kumaraswamy Distribution and moment estimators. Mustansiriyah Journal of pure and Applied Siences, 15,993-1010.," vol. 2, pp. 993–1010, 2009.

[5] M. AA and R. M. Refaey, "Generalized Inverted Kumaraswamy Distribution Properties and Estimation," *المجلة العلمية لقطاع كليات التجارة بجامعة الأزهر*, vol. 24, no. 1, pp. 99–132, 2020.

[6] K. Karakaya, İ. KINACI, K. U. Ş. Coşkun, and Y. Akdoğan, "On the DUS-Kumaraswamy distribution," *Istat. J. Turkish Stat. Assoc.*, vol. 13, no. 1, pp. 29–38, 2021.

[7] A. I. Ishaq *et al.*, "Log-Kumaraswamy distribution: its features and applications," *Front. Appl. Math. Stat.*, vol. 9, p. 1258961, 2023.

[8] M. R. Mahmoud and A. M. Saad, "Estimation of Kumaraswamy Distribution Parameters Using the Principle of Maximum Entropy".

[9] A. M. Abd AL-Fattah, A. A. El-Helbawy, and G. R. Al-Dayian, "INVERTED KUMARASWAMY DISTRIBUTION: PROPERTIES AND ESTIMATION.," *Pakistan J. Stat.*, vol. 33, no. 1, 2017.

[10] M. A. M. Salih and J. H. Eidi, "Using Simulation to Estimate Reliability for Transmuted Inverse Exponential Distribution," in *2019 First International Conference of Computer and Applied Sciences (CAS)*, IEEE, 2019, pp. 16–21.

[11] P. Nasiri and F. G. Masoumi, "Two-Parameter Exponential Distribution with Randomly Censored and Outlier Data," *Jordan J. Math. Stat.*, vol. 17, no. 3, 2024.

[12] C. A. Cohen and B. Whitten, "Modified maximum likelihood and modified moment estimators for the three-parameter Weibull distribution," *Commun. Stat. Methods*, vol. 11, no. 23, pp. 2631–2656, 1982.

[13] A. Zaka and A. S. Akhter, "Modified moment, maximum likelihood and percentile estimators for the parameters of the power function distribution," *Pakistan J. Stat. Oper. Res.*, pp. 369–388, 2014.

[14] N. B. Abbas and A. L. Kneehr, "Comparison between least squares and percentile methods in estimating Rayleigh Kumaraswamy distribution, a simulation study," *Al Kut J. Econ. Adm. Sci.*, vol. 15, no. 48, pp. 561–583, 2023.

[15] A. H. Khaleel and N. S. Karam, "Estimating the reliability function of (2+ 1) cascade model," *Baghdad Sci. J.*, vol. 16, no. 2, pp. 395–402, 2019.

[16] M. A. M. Saleh, "Simulation of the methods of the scale parameter and the reliability function Estimation for two parameters Weibull distribution," *Dr. Philos. Math. Coll. Educ. Al-Mustansiriyah Univ.*, 2006.

[17] M. A. M. Salih and M. M. Hasan, "Linear Formula Estimators For The Reliability Of Transmuted Pareto Distribution," *J. Coll. Educ.*, vol. 2, 2021.

[18] Z. S. Noori and M. A. M. Salih, "Linear Formulas for Estimating the Reliability of Generalized Inverse Weibull Distributi," *Mustansiriyah J. Pure Appl. Sci.*, vol. 1, no. 1, pp. 140–154, 2023.

[19] M. Kayid and M. A. Alshehri, "Modified Maximum Likelihood Estimation of the Inverse Weibull Model," *Axioms*, vol. 12, no. 10, p. 961, 2023.

[20] T. Moakofi, B. Oluyede, A. Puoetsile, and G. Warahena-Liyanage, "A New Generalized Family of Weibull-Exponentiated Half Logistic-G Distribution with Applications," *Cent. Eur. J. Econ. Model. Econom.*, pp. 125–189, 2024.