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Stability of Composition Caputa– Katugampola Fractional Differential Nonlinear Control System with Delay Riemann −Katugampola

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Abstract

This work used the Composition Caputo-Katugampola Fractional Derivatives technique to tackle nonlinear problems and delay fractional differential equations. The fractional derivative is defined using the Caputo and Riemann-Katugampola Fractional Derivatives Method. Proposed method in comparison to other digital technologies, this one is simple, effective, and is not complicated. Ensuring authenticity and correctness of to proposed method by discussing some exemplary problems .

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1. Introduction

Because of its wide applications in simulating diverse physical processes in engineering and research, fractional calculus is employed in many fields of mathematics. The idea of fractional calculus stems from the fact $D^{\alpha}(f(x))$, where alpha is a noninteger. Later on, experts like Riemann-Liouville, Euler, Leibniz, L'Hospital, Bernoulli, and Wallis dedicated their attention to this field. Fractional calculus has many applications in several fields of research. For example, viscoelastic material of dynamics ,electromagnetism [1],[2] fluid mechanics [3], spherical flame propagation [4], and viscoelastic materials[5]. DEs (Differential Equations) are employed in real life to create a variety of health ailments. Some of these differential equations are more complicated and cannot be described by using basic Differential equations methods. The researchers employed a novel fractional differential equations approach to solve these complicated issues. In the mathematical modeling of real-world physical problems, FDEs have been widespread due to their numerous applications in engineering and real-life science problems [6],[7],[8],[6][9] such as economics [10], solid mechanics [11], continuum and statistical mechanics [12], oscillation of earthquakes [13],algebraic [14],network [15] ,dynamics of interfaces between soft-nanoparticles and rough substrates [16], fluid dynamic traffic model [17], colored noise [18], anomalous transport [19], and bioengineering[5],[20]. (delay differential equations (DDEs) have several applications in engineering. The delay differential equation simplifies the ordinary differential equation, Which is data-dependent, and applies to physical systems. Researchers now pay more attention to FDDEs than DEs since a small delay has a huge effect.

Numerous articles have been dedicated to the study of the numerical solution of FDDEs in this respect. FDDs are commonly used in mathematical modeling such as population[21],[22],[23]. Dynamics, epidemiology, immunology, physiology, and neural networks in the brain are every field of study. There is no specific approach in the literature for obtaining an accurate or analytical solution for every FDDE; the researcher aims to find the numerical solution of FDDEs. Various approaches for numerically addressing these issues have been implemented. Among these approaches are the new predictor-corrector method (NPCM) [24], A domain decomposition method (ADM) [25], and others. Legendre pseudo-spectral method (LSM) [26], kernel method (KM) [27], LMS method (LMSM)[28], extended predictor-corrector method (EPCM)[29], For the analytical and numerical solution of FDDEs, the homotopy perturbation method (HPM) [30], Runge-Kutta-type methods (RKM) [3], Bernoulli wavelet method (BWM)[31], and the modified Laguerre wavelet method[32] , have been utilized. Composite (CK) expanded in this paper for the solution of FDDEs. (e findings obtained are compared

with other approaches, demonstrating that CPM has a higher convergence rate than other methods. We concentrate on the form of FDDE.
The structure of the paper is Organized as follows. Section 2 introduces some essential fractional calculus definitions and mathematical procedures that will come into use later in our study. Section 3 obtains existence, uniqueness, and stability theorems. Section 4 illustrative examples of numerical findings that help to understand the process.

We use the steps technique and the extended Gronwall inequality in this paper to develop a sufficient condition for the finite-time stability of nonlinear fractional-order delay System of the form.

$$\begin{cases} {}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}x(t)\right) = \sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) \\ \qquad +Bu(t), \quad t \in [0.T] \\ x(t) = \int_0^t f(s)ds, \qquad t \in [-\tau, 0] \end{cases} \quad 1.1$$

Where ${}^{CK}_{a}D_t^{\alpha,\mu}$ denotes the Caputo −Katugampola Fractional Derivatives of order $\alpha, \mu > 0$ ${}^{RK}_{a}D_t^{\alpha,\mu}$ denotes the Riemann–Katugampola derivative of order $\alpha > 0$, , $\mu > 0$ and $x(.) \in R^n$ for t∈ $[-\infty$,T] , where $0 < a < b < \infty$, ${}^{RK}_{a}D_t^{\alpha,\mu}$ denotes the Riemann −Katugampola Fractional derivative of order $0 < \alpha < 1, \mu$ be a positive value . $f(.,.,.) : [0.T] \times R^n \times R^n \to R^n$, $B_{n\times m}$ is a control matrix and $u(t) : [0, T] \to R^m$ is a control function. Finely the function $x(t)$ is a nonlocal function defined on $[-\tau, 0]$, $\tau \in (-\infty, 0)$.

2. Preliminaries

This section discusses certain definitions and lemmas that related to fractional calculus.

Definition (2.1),[33]**:**

Let $\alpha > 0, \mu > 0$,the set of functions defined on the interval [a,b] that have a Lebesgue integral these functions may not be continuous or even limited, but they must meet certain integrability constraints in order to have a Lebesgue integral($x \in L^1([a,b]), R)$, $0 < a < b < \infty$.)The left and right R-K FDs of order α is defined by

$${}^{RK}_{a}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu - \tau^\mu)^\alpha}\, x(\tau)\, d\tau$$

$${}^{RK}_{b}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu - t^\mu)^\alpha}\, x(\tau)\, d\tau.$$

Definition (2.2),[33]

Let $\alpha > 0, \mu > 0,\ x \in L^1([a,b]), R),\ 0 < a < b < \infty$ .The left and right Riemann−Katugampola fractional integrals (R-KFIs) of order α is defined by

$${}^{RK}_{a}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{\alpha-1} x(\tau)\, \frac{d\tau}{\tau^{1-\mu}}$$

$${}^{RK}_{b}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{\tau^\mu - t^\mu}{\mu}\right)^{\alpha-1} x(\tau)\, \frac{d\tau}{\tau^{1-\mu}}.$$

Definition (2.3),[33]**:**

Let $\alpha \in (0,1), \mu > 0, [a,b] \in R, 0 < a < b < \infty$, The left and right C–KFDs are defined respectively by

$${}^{CK}_{a}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu - \tau^\mu)^\alpha}\, (x(\tau) - x(a))\, d\tau,$$

$${}^{CK}_{b}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu - t^\mu)^\alpha}\, (x(\tau) - x(b))\, d\tau.$$

Lemma (2.4),[34]**:**

Assume $0 < g \le f, \alpha \in (0,1]$. Then $f^\alpha - g^\alpha \le (f-g)^\alpha$

3.Problem Formulation for Proposal system

This section discusses the existence, uniqueness, and stability theorems for -order nonlinear differential equations.

Theorem (3.5):

Let $\alpha > 0$, then the equality ${}^{RK}_{a}D^{\alpha,\mu}_{t}\left({}^{RK}_{a}D^{-\alpha,\mu}_{t}x\right) = x(t)$ belongs to the space $L_1(a,b)$ if its absolute value $|x(t)|$ is integrable over the interval (a,b) with respect to the Lebesgue measure($x(t) \in L_1(a,b)$),$0 < \alpha < 1$.

Proof :

Form definition (2.1) and (2.2), we have that

$${}^{RK}_{a}D^{\alpha,\mu}_{t}\left({}^{RK}_{a}D^{-\alpha,\mu}_{t}x\right) = \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{s^{\mu-1}ds}{(t^{\mu}-s^{\mu})^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_a^s \left(\frac{s^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1} x(\tau)\,\frac{d\tau}{\tau^{1-\mu}}\right)$$

$$= \frac{\mu^{\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{s^{\mu-1}ds}{(t^{\mu}-s^{\mu})^{\alpha}}\left(\int_a^s \left(\frac{s^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1} x(\tau)\,\frac{d\tau}{\tau^{1-\mu}}\right)$$

$$= \frac{\mu^{1-\alpha}\mu^{\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{s^{\mu-1}ds}{(t^{\mu}-s^{\mu})^{\alpha}}\left(\int_a^s \frac{x(\tau)}{(s^{\mu}-\tau^{\mu})^{1-\alpha}}\,\frac{d\tau}{\tau^{1-\mu}}\right)$$

$$= \frac{\mu}{\Gamma(\alpha)\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}x(\tau)d\tau}{(t^{\mu}-\tau^{\mu})}$$

$$= \frac{1}{\Gamma(1)}(-t^{1-\mu}\frac{d}{dt})\int_a^t \frac{-\mu\tau^{\mu-1}x(\tau)d\tau}{(t^{\mu}-\tau^{\mu})}$$

$$= (-t^{1-\mu}\frac{d}{dt})\int_a^t \frac{-\mu\tau^{\mu-1}x(\tau)d\tau}{(t^{\mu}-\tau^{\mu})}$$

$$= x(t)$$

Lemma (3.6),[35]:

Let f be a continuous function on a rectangle $R = [a,b]\,x\,[c,d]$, then $\int\int f(x,y)d(x,y)$ have the following $\int_a^b(\int_c^d f(x,y)dy)dx = \int_c^d(\int_a^b f(x,y)dx)dy.$

Theorem (3.7):

Let $\alpha > 0, \beta > 0, 1 \le p \le \infty, 0 < a < b < \infty$ and let $\mu \in R$ and $c \in R$ be such that $\mu \ge c$. Then for $x \in X^p_c(a,b)$ the semi group property holds. That is ${}^{RK}_{a}D^{-\alpha,\mu}_{t}\left({}^{RK}_{a}D^{-\beta,\mu}_{t}x\right) = {}^{RK}_{a}D^{-\alpha-\beta,\mu}_{t}x(t)$

Proof:

By using lemma (3.6), we have that

$${}^{RK}_{a}D^{-\alpha,\mu}_{t}\left({}^{RK}_{a}D^{-\beta,\mu}_{t}x(\tau)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}\left(\int_a^{\tau}\left(\frac{\tau^{\mu}-s^{\mu}}{\mu}\right)^{\beta-1}\frac{x(s)}{s^{1-\mu}}ds\right)\frac{x(\tau)}{\tau^{1-\mu}}d\tau$$

Now we have that,

$${}^{RK}_{a}D^{-\alpha,\mu}_{t}\left({}^{RK}_{a}D^{-\beta,\mu}_{t}x(\tau)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left(\int_{\tau}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}\left(\frac{\tau^{\mu}-s^{\mu}}{\mu}\right)^{\beta-1}\frac{x(s)}{s^{1-\mu}}ds\right)\frac{1}{\tau^{1-\mu}}d\tau$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_\tau^t\left(\frac{t^\mu-s^\mu}{\mu}\right)^{\alpha+\beta-2}\frac{x(s)}{s^{1-\mu}}ds\right)\frac{1}{\tau^{1-\mu}}d\tau$$

$$=\frac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha+\beta-1}\frac{x(\tau)}{\tau^{1-\mu}}\right)d\tau$$

$$=\frac{1}{\Gamma(\alpha+\beta)}\int_a^t\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{-\alpha-\beta-1}\frac{x(\tau)}{\tau^{1-\mu}}dt$$

$${}_a^{RK}D_t^{-\alpha-\beta,\mu}x(\tau)$$

Lemma (3.8),[33]**:**

$f(x)$ belongs to the space $C^n[0,+\infty]$ if $f(x)$ and its first n derivatives $f'(x), f''(x), \dots, f^n(x)$ are all continuous on the interval [0,+∞).

If $f(t)\in C^n[0,+\infty]$and $n-1<\alpha<n\in z^+$,

1. ${}_a^{RK}D_t^{\alpha,\mu}({}_a^{RK}D_t^{-\alpha,\mu}f(t))=f(t)$
2. ${}_a^{CK}D_t^{\alpha,\mu}f(t)={}_a^{RK}D_t^{\alpha,\mu}f(t)-\frac{\mu^\alpha f(a)}{\Gamma(1-\alpha)}(t^\mu-a^\mu)^{-\boldsymbol{\alpha}}$

Lemma (3.9):

The relation between Riemann−Katugampola fractional integrals and Composition Caputo −Katugampola fractional derivatives have the following formulation ${}_a^{RK}D_t^{-\alpha-\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)\right)\right)=f(t)$

Proof :

Form definition (2.2) and (2.3), we have that

$${}_a^{RK}D_t^{-\alpha-\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)\right)\right)$$

$$={}_a^{RK}D_t^{-\alpha-\alpha,\mu}\left({}_a^{RK}D_t^{\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)\right)-\frac{\mu^\alpha(t^\mu-a^\mu)^{-\alpha}}{\Gamma(1-\alpha)}{}_a^{CK}D_t^{\alpha,\mu}f(a)\right)$$

$$={}_a^{RK}D_t^{-\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)-{}_a^{RK}D_t^{-\alpha,\mu}\left(\frac{\mu^\alpha(t^\mu-a^\mu)^{-\alpha}}{\Gamma(1-\alpha)}{}_a^{CK}D_t^{\alpha,\mu}f(a)\right)\right)$$

$$={}_a^{RK}D_t^{-\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)-\frac{\mu^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_a^a\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha-1}{}_a^{CK}D_t^{\alpha,\mu}f(\tau)(t^\mu-\tau^\mu)^{-\alpha}\tau^{\mu-1}d\tau\right)$$

Since $f(t)=f(a)$ then $t=a$ therefore the integration in [a,a] equal zero

$${}_a^{RK}D_t^{-\alpha-\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}\left({}_a^{CK}D_t^{\alpha,\mu}f(t)\right)\right)=f(t)$$

Lemma (3.10):

Suppose $\alpha>0$, $a(t)$ is a nonnegative function locally integrable on $0\le t\le T$ (some $T\le+\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0\le t\le T$ $g(t)\le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0\le t\le T$ with

$$u(t)=a(t)+g(t)\int_0^t\left(\frac{(t^\mu-s^\mu)}{\mu}\right)^{\alpha-1}u(s)\,ds$$

on this interval. Then

$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\frac{(t^{\mu} - s^{\mu})}{\mu}\right)^{n\alpha-1} \; a(s)\right] ds$ or

$u(t) \leq a(t) E_{\alpha}\left(g(t) \left(\frac{t^{\alpha\mu}}{\mu}\right)\right)$

proof :

Let $B\varphi(t) = \frac{g(t)}{\Gamma(2\alpha)} \int_a^t \left(\frac{(t^{\mu} - s^{\mu})}{\mu}\right)^{2\alpha-1} \; \frac{x(\tau)}{\tau} d\tau \; , t \geq \; 0$, for locally integrable functions φ. Then $u(t) \leq a(t) + Bu(t)$ imply

$$u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t)$$

Now to prove $B^n u(t) \leq \int_a^t \frac{(g(t)\Gamma(2\alpha))^n}{\Gamma(2n\alpha)} \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{2n\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\mu}}$ 3.2

$$B^n u(t) \to 0 \; as \; n \longrightarrow +\infty \quad \forall t \in [0, T)$$

We know this relation (3.2) is true for n = 1. Assume that it is true for some n = k. If n = k +1, then the induction hypothesis implies

$B^{k+1}u(t) = B(B^k) \leq \frac{g(t)}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^{\mu}-s^{\mu}}{\mu}\right)^{2\alpha-1} \left[\int_a^s \frac{(g(s)\Gamma(2\alpha))^k}{\Gamma(2k\alpha)} \left(\frac{s^{\mu}-\tau^{\mu}}{\mu}\right)^{2k\alpha-1} u(\tau) d\tau\right] \frac{ds}{s^{1-\mu}}$

Since *g(t)* is nondecreasing, it follows that

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^{\mu}-s^{\mu}}{\mu}\right)^{2\alpha-1} \left[\int_a^s \frac{(\Gamma(2\alpha))^k}{\Gamma(2k\alpha)} \left(\frac{s^{\mu}-\tau^{\mu}}{\mu}\right)^{2k\alpha-1} u(\tau) d\tau\right] \frac{ds}{s^{1-\mu}}$

By interchanging the order of integration, we have

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(2\alpha))^k}{\Gamma(2k\alpha)} \left(\frac{t^{\mu}-s^{\mu}}{\mu}\right)^{2\alpha-1} \left(\frac{s^{\mu}-\tau^{\mu}}{\mu}\right)^{2k\alpha-1} ds\right] u(\tau) \frac{ds}{s^{1-\mu}}$

Where $z \in [0, T]$, β is a beta function

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(2\alpha))^n}{\Gamma(2n\alpha)} \left(\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)(1-z)\right)^{2\alpha-1} \left(\frac{z(t^{\mu} - \tau^{\mu})}{\mu}\right)^{2k\alpha-1} \quad ds\right] u(\tau) \frac{d\tau}{\tau^{1-\mu}}$

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_{\tau}^t \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1} \left(\frac{(t^{\mu} - \tau^{\mu})}{\mu}\right)^{k\alpha-1} (1-z)^{\alpha} z^{k\alpha-1} \quad dz\right] u(\tau) \frac{d\tau}{\tau^{1-\mu}}$

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left[\int_0^1 \frac{(\Gamma(2\alpha))^k}{\Gamma(k\alpha)} \left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{2\alpha-1} \left(\frac{(t^{\mu} - \tau^{\mu})}{\mu}\right)^{2k\alpha-1} (1-z)^{2\alpha} z^{2k\alpha-1} \quad dz\right] u(\tau) \frac{d\tau}{\tau^{1-\mu}}$

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\frac{(t^{\mu} - \tau^{\mu})}{\mu}\right)^{k\alpha+\alpha-1} \int_0^1 (1-z)^{\alpha} z^{k\alpha-1} \quad dz\right] u(\tau) \frac{d\tau}{\tau^{1-\mu}}$

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left[\frac{(\Gamma(2\alpha))^k}{\Gamma(2k\alpha)} \left(\frac{(t^{\mu} - s^{\mu})}{\mu}\right)^{2(k+1)\alpha-1} B(k\alpha, \alpha)\right] u(s) \frac{ds}{s^{1-\mu}}$

$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(2\alpha)} \int_a^t \left[\frac{(\Gamma(2\alpha))^k}{\Gamma(2k\alpha)} \left(\frac{(t^{\mu} - s^{\mu})}{\mu}\right)^{2(k+1)\alpha-1} \frac{\Gamma(2\alpha)\Gamma(2n\alpha)}{\Gamma(2(k+1)\alpha)}\right] u(s) \frac{ds}{s^{1-\mu}}$

$$B^{k+1}u(t) \leq \int_a^t \left[\frac{\left(g(t)\Gamma(2\alpha)\right)^{k+1}}{\Gamma(2\alpha)\Gamma(2(k+1)\alpha)}\left(\frac{(t^{\mu}-s^{\mu})}{\mu}\right)^{2(k+1)\alpha-1}\right]u(s)\frac{ds}{s^{1-\mu}}$$

Since $B^n u(t) \leq \int_a^t \left[\frac{\left(g(t)\Gamma(\alpha)\right)^{n}}{\Gamma(2\alpha)\Gamma(2(k+1)\alpha)}\left(\frac{(t^{\mu}-s^{\mu})}{\mu}\right)^{2n\alpha-1}\right]u(s)ds \to 0, n \to \infty\ \forall t \in [0,T)$

$$u(t) \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^{n}}{\Gamma(2\alpha)\Gamma(2n\alpha)}\left(\frac{(t^{\mu}-s^{\mu})}{\mu}\right)^{2n\alpha-1} a(s)\right]\frac{ds}{s^{1-\mu}} \qquad 0 \leq t \leq T$$

$$u(t) \leq a(t)\left(1+\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(2\alpha)\right)^{n}}{\Gamma(2\alpha)\Gamma(2n\alpha)}\int_a^t\left[\left(\frac{(t^{\mu}-s^{\mu})}{\mu}\right)^{2n\alpha-1}\right]\frac{ds}{s^{1-\mu}}\right)$$

$$u(t) \leq a(t)\left(1+\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^{n}}{\Gamma(2\alpha)(2n\alpha)\Gamma(2n\alpha)}\left(\frac{(t^{\mu}-s^{\mu})}{\mu}\right)^{2n\alpha}\Bigg|_a^t\right)$$

$$u(t) \leq a(t)\left(1+\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(2\alpha)\right)^{n}}{\Gamma(2\alpha)(2n\alpha)\Gamma(2n\alpha)}\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2n\alpha}\right)$$

$$u(t) \leq a(t)\left(1+\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(2\alpha)\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2\alpha}\right)^{n}}{\Gamma(2\alpha)(2n\alpha)\Gamma(2n\alpha)}\right)$$

$$u(t) \leq a(t)\left(1+\frac{1}{\Gamma(2\alpha)}\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(2\alpha)\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2\alpha}\right)^{n}}{\Gamma(2n\alpha+1)}\right)$$

$$u(t) \leq a(t)\left(\frac{1}{\Gamma(2\alpha)}\sum_{n=0}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2\alpha}\right)^{n}}{\Gamma(2n\alpha+1)}\right)$$

$$u(t) \leq a(t)E_{\alpha}\left(g(t)\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2\alpha}\right).$$

Lemma (3.11) :

Let $\alpha > 0, \mu > 0$,and $x(t) \in L^1([a,b]), R)$, $0 < a < b < \infty$ then $(\left\|\left({}^{RK}_{a}D_t^{\alpha,\mu}\right)x(t) - \left({}^{RK}_{a}D_t^{\alpha,\mu}\right)y(t))\right\| \leq C_1(t)\|x(\tau)-y(\tau)\|$ when $C_1(t) = \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left((t^{\mu}-a^{\mu})\right)$

Proof:

$$\left(\left({}^{RK}_{a}D_t^{\alpha,\mu}\right)x(t) - \left({}^{RK}_{a}D_t^{\alpha,\mu}\right)y(t)\right) = \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{(1-\mu)}\frac{d}{dt}\right)\int_0^t \frac{\tau^{(\mu-1)}}{(t^{\mu}-\tau^{\mu})^{\alpha}}x(\tau)d\tau - \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{(1-\mu)}\frac{d}{dt}\right)\int_a^t \frac{\tau^{(\mu-1)}}{(t^{\mu}-\tau^{\mu})^{\alpha}}y(\tau)d\tau$$

$$= \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^{\mu}-\tau^{\mu})^{\alpha}}\left(x(\tau)-y(\tau)\right)d\tau\right]$$

Therefore

$$\left\|\left({}^{RK}_{a}D_t^{\alpha,\mu}\right)x(t) - \left({}^{RK}_{a}D_t^{\alpha,\mu}\right)y(t)\right\| \leq \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^{\mu}-\tau^{\mu})^{\alpha}}\|x(\tau)-y(\tau)\|d\tau\right]$$

$$\left\|\left({}^{RK}_{a}D_t^{\alpha,\mu}\right)x(t) - \left({}^{RK}_{a}D_t^{\alpha,\mu}\right)y(t)\right\| \leq \|x(t)-y(t)\|\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^{\mu}-\tau^{\mu})^{\alpha}}d\tau\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{-\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_{a}^{t} -(t^{\mu}-\tau^{\mu})^{-\alpha}\mu\tau^{(\mu-1)}d\tau\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{-\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\left.\frac{(t^{\mu}-\tau^{\mu})^{-\alpha+1}}{-\alpha+1}\right|_{a}^{t}\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{-\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{0-(t^{\mu}-a^{\mu})^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{(t^{\mu}-a^{\mu})^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{(t^{\mu}-a^{\mu})^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \|x(t)-y(t)\| \frac{\mu^{\alpha}}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{(-\alpha+1)(t^{\mu}-a^{\mu})^{-\alpha}\mu t^{\mu-1}}{-\alpha+1}\right)$$

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left((t^{\mu}-a^{\mu})\right)\|x(t)-y(t)\|$$

set $C_1(t) = \frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left((t^{\mu}-a^{\mu})\right)$, we get

$$\left\|\left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)x(t) - \left({}^{RK}_{a}D^{\alpha,\mu}_{t}\right)y(t)\right\| \leq C_1(t)\|x(t)-y(t)\| \quad 3.3$$

Lemma (3.12):

Let $\alpha > 0, \mu > 0$,and $x(t) \in L^1([a,b]),R)$, $0 < a < b < \infty$ $\left\|{}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t)\right\| \leq C_2(t)\|x(t) - y(t)\|$ where $C_2(t) = \frac{1}{\mu\alpha\Gamma(\alpha)}\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)$

Proof:

$${}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1} x(\tau)\,\frac{d\tau}{\tau^{1-\mu}} - \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1} y(\tau)\,\frac{d\tau}{\tau^{1-\mu}}$$

$$= \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}(x(\tau)-y(\tau))\,\frac{d\tau}{\tau^{1-\mu}}$$

Therefore

$$\left\|{}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t)\right\| \leq \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}\|x(\tau)-y(\tau)\|\,\frac{d\tau}{\tau^{1-\mu}}$$

$$\left\|{}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t)\right\| \leq \|x(t) - y(t)\|\frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}\tau^{\mu-1}d\tau$$

$$\left\|{}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t)\right\| \leq \|x(t) - y(t)\|\frac{-1}{\mu\Gamma(\alpha)}\int_{a}^{t}\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}(-\tau^{\mu-1})d\tau$$

$$\left\|{}^{RK}_{a}D^{-\alpha,\mu}_{t}x(t) - {}^{RK}_{a}D^{-\alpha,\mu}_{t}y(t)\right\| \leq \|x(t) - y(t)\|\frac{-1}{\mu\Gamma(\alpha)}\left.\frac{\left(\frac{t^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha}}{\alpha}\right|_{a}^{t}$$

$$\left\| {}^{RK}_{a}D_t^{-\alpha,\mu}x(t) - {}^{RK}_{a}D_t^{-\alpha,\mu}y(t)\right\| \le \|x(t) - y(t)\| \frac{-1}{\mu\alpha\Gamma(\alpha)}\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^\alpha\Bigg|_a^t$$

$$\left\| {}^{RK}_{a}D_t^{-\alpha,\mu}x(t) - {}^{RK}_{a}D_t^{-\alpha,\mu}y(t)\right\| \le \|x(t) - y(t)\| \frac{-1}{\mu\alpha\Gamma(\alpha)}\left(-\frac{t^\mu-a^\mu}{\mu}\right)$$

set $C_2(t) = \frac{1}{\mu\alpha\Gamma(\alpha)}\left(\frac{t^\mu-a^\mu}{\mu}\right)$ we get

$$\left\| {}^{RK}_{a}D_t^{-\alpha,\mu}x(t) - {}^{RK}_{a}D_t^{-\alpha,\mu}y(t)\right\| \le C_2(t)\|x(t) - y(t)\| \qquad 3.4$$

Theorem (3.13):

Let $x:[-\tau,T] \to R^m$ be continuous differential function , $\tau > 0$ then x(t) is a solution of the Composition Caputo −Katugampola fractional order nonlinear differential control nonlocal system (1.1),if and only if

$$x(t) = \begin{cases} {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right) & \\ \int_0^t f(s)ds & for -\tau \le t \le 0 \end{cases}$$

3.5

Proof:

For $-\tau \le t \le 0$, we have the solution is $x(t) = \int_0^t f(s)ds$, now from (3.5) , we have that

$x(t) = {}^{RK}_{a}D_t^{-\alpha-\alpha,\mu}\left(\sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right)$ $\quad 0 \le t \le T$, implies that

, ${}^{RK}_{a}D_t^{-\alpha-\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{-\alpha,\mu}x(t)\right)\right) = {}^{RK}_{a}D_t^{-\alpha-\alpha,\mu}\left(\sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right)$ using lemma (3.7), we obtain

${}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{-\alpha,\mu}x(t)\right) = \sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)$ .The necessity of the condition (3.5)

The solution for system (1.1) is $x(t) = \int_0^t f(s)ds$ for $-\tau \le t \le 0$

${}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}x(t)\right) = \sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)$

${}^{RK}_{a}D_t^{-\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}x(t)\right)\right) = {}^{RK}_{a}D_t^{-\alpha,\mu}\left(\sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right)$ ${}^{RK}_{a}D_t^{-\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu}x(t)\right) = {}^{RK}_{a}D_t^{-\alpha-\alpha,\mu}\left(\sum_{I=1}^n A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right)$

$x(t) = {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t)\right)$ is the solution of (1.1).

Theorem (3.4):

Consider the following Composition Caputo −Katugampola fractional order nonlinear differential control nonlocal system and ${}^{CK}_{a}D_t^{\alpha,\mu}\left({}^{CK}_{a}D_t^{\alpha,\mu} x(t)\right) = \sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t)$ with $(f(t,0,0) = [0,\dots,0]^T)$ has a unique continuous solution

proof:

Let $x(t)$ and $y(t)$ be any two different solutions to system (1.1), then $x(t)$ and $y(t)$ both satisfy the formulation of solution (3.5)

Also, let $\xi(t) = x(t) - y(t) = \int_0^t f(s)ds - \int_0^t f(s)ds = 0$, one can obtain $\xi(t) = 0\, for\ -\tau \le t \le 0$. Hence the system (1.1) has a unique continuous solution for $-\tau \le t \le 0$.

As well as for $0 \le t \le T$, we have then

$$\xi(t) = x(t) - y(t) = {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t)\right) - {}^{RK}_{a}D_t^{-\alpha-\alpha,\mu}\left(f \sum_{I=1}^{n} A_i y(t) + By(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} y(t-\tau), y(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} y(t), y(t)\right) + Bu(t)\right)$$

$$\xi(t) = {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^{n} A_i \xi(t) + B\xi(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(t-\tau), \xi(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(t), \xi(t)\right)\right)$$

$$\xi(t) = \frac{1}{\Gamma(\alpha+\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \left(\left(\sum_{I=1}^{n} A_i \xi(\tau) + B\xi(\tau-s) + f\left(\tau, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau-s), \xi(\tau-s)\right) g\left(\tau, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau), \xi(\tau)\right)\right)\right) \frac{d\tau}{\tau^{1-\mu}}$$

Where $0 \le t \le \tau$, we have that

$$\xi(t) = \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^{n} A_i \xi(\tau) + B\xi(\tau-s) + f\left(\tau, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau-s), \xi(\tau-s)\right) g\left(\tau, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau), \xi(\tau)\right)\right) \frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi(t)\| \le \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^{n} \|A_i\| \|\xi(\tau)\| + \|B\xi(\tau-s)\| + h\left(\tau, {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau-s), \xi(\tau-s), {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau), \xi(\tau)\right) - h(\tau,0,0,0,0)\right) \frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi(t)\| \le \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^{n} \|A_i\| \|\xi(\tau)\| + \|B\xi(\tau-s)\| + L\left({}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau-s) + \xi(\tau-s) + {}^{RK}_{a}D_t^{\alpha,\mu} \xi(\tau) + \xi(\tau)\right)\right) \frac{d\tau}{\tau^{1-\mu}} \text{ by (3.3),(3.4)}$$

$\|\xi(t)\| \leq \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^n \|A_i\| \|\xi(\tau)\| + \|B\xi(\tau-s)\| + \left\|L\left(C_1(\tau)\xi(\tau-s) + \xi(\tau-\tau) + C_1(\tau)\xi(\tau) + \xi(\tau)\right)\right\|\right)$

$\|\xi(t)\| \leq \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^n \|A_i\| \|\xi^*(t)\| + \|B\xi^*(t-\tau)\| + \left\|L\left(C_1(t)\xi^*(t-\tau) + \xi^*(t-\tau) + C_1(t)\xi^*(t) + \xi^*(t)\right)\right\|\right)$

$\|\xi(t)\| \leq \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \left(\sum_{I=1}^n \|A_i\| \|\xi^*(t)\| + \|B\xi^*(t-\tau)\| + \left\|L\left(C_1(t)\xi^*(t-\tau) + \xi^*(t-\tau) + C_1(t)\xi^*(t) + \xi^*(t)\right)\right\|\right) \frac{d\tau}{\tau^{1-\mu}}$

Where $\xi^*(t) = \sup_{\vartheta\in[-\tau,0]} \|\xi(t+\vartheta)\|$

$\|\xi(t)\| \leq \left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+1+C_1(t)+1)\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$

$\|\xi(t)\| \leq \left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$

$\|\xi^*(t)\| \leq \left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(\tau)\| \frac{d\tau}{\tau^{1-\mu}}$

Set g(t)= $\left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right)$

$$\|\xi^*(t)\| \leq \mathrm{g(t)} \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(\tau)\| \frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi^*(t)\| \leq 0E_\alpha\left(\mathrm{g(t)}\left(\frac{\mathrm{t}^\mu - \mathrm{a}^\mu}{\mu}\right)^{2\alpha\mu}\right)$$

$\|\xi^*(t)\| \leq 0$

$\|\xi^*(t)\| \leq x(t) - y(t) \leq 0$

$x(t) = y(t)$

If $\tau \leq t < T$ $then$ $\xi(t) = x(t) - y(t)$

$$\|\xi^*(t)\| \leq \frac{\left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right)}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$$

$$when\ \hat{L}_1 = \left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right)$$

$\|\xi^*(t)\| \leq \left(\sum_{I=1}^n \|A_i\| + \|B\| + \|L(C_1(t)+2+C_1(t))\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$

$$\|\xi^*(t)\| \leq \hat{L}_1 \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$$

$\|\xi^*(t)\| \leq \hat{L}_1 \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$

$\|\xi^*(t)\| \leq \left(\hat{L}_1\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1} \|\xi^*(t)\| \frac{d\tau}{\tau^{1-\mu}}$

$\|\xi^*(t)\| \leq 0E_\alpha\left(\hat{L}_1\right) \left(\frac{t^\mu-a^\mu}{\mu}\right)^{2\alpha-1}$

$\|\xi^*(t)\| \leq 0$,Then $x(t) - y(t) = 0$

Therefore $x(t) = y(t)$

Then system (1.1) has a unique continuous solution .

3.1 Stability of the Composition Caputa– Katugampola and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system with maximal interval (0,T]

Theorem (3.1.15):

The solution to the Composition Caputa −Katugampola fractional derivatives Riemann−Katugampola fractional order nonlinear differential control nonlocal system (1.1) is $x(t) \in R^n$ Then the following inequalities holds:

$$\| x(t) \| \leq \delta(t) E_\alpha \left(\widehat{\omega}(t) \left(\frac{t^\mu - a^\mu}{\mu} \right)^{2\alpha} \right) \qquad 3.6$$

Proof:

Since $x(t)$ has the following formulation,

$$x(t) = \begin{cases} {}^{RK}_{a}D_t^{-2\alpha,\mu} \left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t) \right) \\ \int_0^t f(s)ds \qquad\qquad t \in [-\tau, 0] \end{cases}$$

. Thus, for $0 \leq t \leq T$, we have

$$x(t) = \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t) \right) \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| = \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \left\| \begin{matrix} \sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) \\ g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t) \end{matrix} \right\| \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| = \frac{1}{\Gamma(2\alpha)} \left\| \begin{matrix} \sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) \\ g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) + Bu(t) \end{matrix} \right\| \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\| \|x(t)\| + \|B\| \|x(t-\tau)\| + \left\| \begin{matrix} f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau)\right) \\ g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) \end{matrix} \right\| + \|B\| \|u(\tau)\| \right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\| \|x(t)\| + \|B\| \|x(t-\tau)\| + \left\| h\left(t, {}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau), x(t-\tau), {}^{RK}_{a}D_t^{\alpha,\mu} x(t), x(t)\right) - h(t,0,0,0,0) \right\| + \|B\| \|u(\tau)\| \right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\| \|x(t)\| + \|B\| \|x(t-\tau)\| + \left\| L\left({}^{RK}_{a}D_t^{\alpha,\mu} x(t-\tau) + x(t-\tau) + {}^{RK}_{a}D_t^{\alpha,\mu} x(t) + x(t)\right) + \right\| + \|B\| \|u(\tau)\| \right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu} \right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$$

$\|x(t)\| \leq (\|\sum_{I=1}^{n} A_i\|\|x(t)\| + \|B\|\|x(t-\tau)\| + L(\|C_1(t)x(t-\tau)\| + \|x(t-\tau)\| + \|C_1(t)x(t)\| + \|x(t)\|) + \|B\|\|u(\tau)\|) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$

$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\|\|x(t)\| + \|B\| \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + L\left(C_1(t)\left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + \|C_1(t)x(t)\| + \|x(t)\|\right) + \|B\|\|u(\tau)\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}}$

$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\|\|x(t)\| + \|B\| \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + L\left(C_1(t)\left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| + \|C_1(t)x(t)\| + \|x(t)\|\right) + \right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}} + \|B\|\|u(\tau)\| \frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}$

$\|x(t)\| \leq \left(\|\sum_{I=1}^{n} A_i\| + \|B\| + L(C_1(t) + 2 + \|C_1(t)\|)\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| \frac{d\tau}{\tau^{1-\mu}} + \|B\|\|u(\tau)\| \frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}$

$Set\ \delta(t) = \|B\|\|u(\tau)\| \frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}$

$$\widehat{\omega}(t) = \left(\left\|\sum_{I=1}^{n} A_i\right\| + \|B\| + L(2C_1(t) + 2)\right)$$

$\|x(t)\| \leq \delta(t) + \widehat{\omega}(t) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{\alpha-1} \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\| \frac{d\tau}{\tau^{1-\mu}}$

$\|x(t)\| \leq \delta(t) + \left(\widehat{\omega}(t)\left({}^{RK}_{\ a}D_t^{-\alpha,\mu} \left\|\sup_{\vartheta\in[-\tau,0]} x(t+\vartheta)\right\|\right)\right)$

$\| x(t) \| \leq \delta(t) E_\alpha \left(\widehat{\omega}(t)\left(\frac{t^\mu - a^\mu}{\mu}\right)^{2\alpha}\right).$

3.2 Stability of the Composition Caputa– Katugampola and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system with by using step method.

Theorem (3.2.16):

Assume that Composition Caputo −Katugampola fractional order nonlinear differential control nonlocal system (1.1) satisfy the following conditions lemma (3.10). Then the solution of (1.1) is finite-time stable, if the following conditions is satisfied: $\omega_T(\tau)E_\alpha\sigma_0(T)\left(\frac{T^\mu}{\mu}\right)^\alpha \leq \varepsilon$ where

$\delta_T(\tau) = \delta_1(T) + \left((C_1(t) + B + 1)\right)\left[\frac{1}{\Gamma(1+2\alpha)}\left(\frac{(j\tau^\mu)}{\mu}\right)^{2\alpha}\left(\sum_{j=1}^{n} \delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\frac{(j\tau^\mu)}{\mu}\right)^{2\alpha}\right) + \frac{1}{\Gamma(1+\alpha)}\left(\frac{(T^\mu) - ((n+1)\tau^\mu)}{\mu}\right)^{2\alpha} \delta_{n+1}(\tau)E_\alpha\sigma_0(n+1)\left(\frac{((n+1)\tau^\mu)}{\mu}\right)^{2\alpha}\right]$

$$\delta_{i+1}(\tau) = \delta_1\big((i+1)\tau\big) + (C_1(t) + B + 1)\left[\frac{1}{\Gamma(1+2\alpha)}\left(\frac{(\tau)^\mu)}{\mu}\right)^{2\alpha}\left[\sum_{j=1}^{i} \delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\frac{\mathrm{j}\tau^\mu}{\mu}\right)^{2\alpha}\right]\right]$$

$\delta_1(t) = (\|B\|\|u(t)\|)\frac{\left(\frac{t^\mu}{\mu}-\frac{a^\mu}{\mu}\right)^{2\alpha}}{\Gamma(1+2\alpha)}$

Proof :

From the system (3.5)

$$x(t) = {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right)$$

$$x(t) = {}^{RK}_{a}D_t^{-2\alpha,\mu}\left(\sum_{I=1}^{n} A_i x(t) + Bx(t-\tau) + f\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t-\tau), x(t-\tau)\right) g\left(t, {}^{RK}_{a}D_t^{\alpha,\mu}x(t), x(t)\right) + Bu(t)\right) \qquad t \in [0,T]$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\|\|x(t)\| + LC_1(t)\|x(t)\| + \|x(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}} + (C_1(t)\|x(t-s)\| + B\|x(t-s)\| + \|x(t-s)\| + \|B\|\|u(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\|\|x(t)\| + LC_1(t)\|x(t)\| + \|x(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}} + \left(C_1(t)\left\|\int_0^t x(s)ds\right\| + B\left\|\int_0^t x(s)ds\right\| + \left\|\int_0^t x(s)ds\right\| + \|B\|\|u(t)\|\right)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\|x(t)\|\frac{d\tau}{\tau^{1-\mu}} + (C_1(t)\left\|\int_0^t x(s)ds\right\| + B\left\|\int_0^t x(s)ds\right\| + \left\|\int_0^t x(s)ds\right\| + \|B\|\|u(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}x(t) + (C_1(t)\left\|\int_0^t x(s)ds\right\| + B\left\|\int_0^t x(s)ds\right\| + \left\|\int_0^t x(s)ds\right\| + \|B\|\|u(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}x(t) + (tC_1(t)\|x(t)\| + Bt\|x(t)\| + t\|x(t)\| + \|B\|\|u(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1 + tC_1(t) + Bt + t)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}x(t) + (\|B\|\|u(t)\|)\frac{1}{\Gamma(2\alpha)}\int_a^t \left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha-1}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le (\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1 + tC_1(t) + Bt + t)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}x(t) + (\|B\|\|u(t)\|)\frac{\left(\frac{t^\mu}{\mu}-\frac{a^\mu}{\mu}\right)^{2\alpha}}{\Gamma(1+2\alpha)}$$

Set $\delta_1(t) = (\|B\|\|u(t)\|)\frac{\left(\frac{t^\mu}{\mu}-\frac{a^\mu}{\mu}\right)^{2\alpha}}{\Gamma(1+2\alpha)}$,

$$\sigma_0(t) = \left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1 + tC_1(t) + Bt + t\right)$$

Using the Lemma (3.10) for $t \in [0,\tau]$, we have that

$\|x(t)\| \le \delta_1(t) + \sigma_0(t)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}x(t)$ for $t \in [0,\tau]$

$$\|x(t)\| \le \delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{t^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}$$

For $t \in (i\tau, (i+1)\tau], 1 \le i \le n$, we have

$$\|x(t)\| \le \left(\left\|\sum_{I=1}^{n} A_i\right\| \|x(t)\| + LC_1(t)\|x(t)\| + \|x(t)\| + \|B\|\|u(\tau)\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha-1} \frac{d\tau}{\tau^{1-\mu}} + \left(C_1(t)\|x(t-s)\| + B\|x(t-s)\| + \|x(t-s)\|\right) \frac{1}{\Gamma(2\alpha)} \int_a^t \left(\frac{t^\mu - s^\mu}{\mu}\right)^{2\alpha-1} \|x(s-\tau)\| \frac{d\tau}{\tau^{1-\mu}}$$

$$\|x(t)\| \le \left(\left\|\sum_{I=1}^{n} A_i\right\| \|x(t)\| + LC_1(t)\|x(t)\| + \|x(t)\| + \|B\|\|u(\tau)\|\right) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{t^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha} + (C_1(t) + B + 1) \left(\frac{1}{\Gamma(2\alpha)} \int_a^\tau \left(\frac{t^\mu - s^\mu}{\mu}\right)^{2\alpha-1} [\|x(s-\tau)\|] \frac{ds}{s^{1-\mu}} + \frac{1}{\Gamma(2\alpha)} \int_\tau^{2\tau} \left(\frac{t^\mu - s^\mu}{\mu}\right)^{2\alpha-1} [\|x(s-\tau)\|] \frac{ds}{s^{1-\mu}} + \cdots + \frac{1}{\Gamma(2\alpha)} \int_{i\tau}^{t} \left(\frac{t^\mu - s^\mu}{\mu}\right)^{2\alpha-1} [\|x(s-\tau)\|] \frac{ds}{s^{1-\mu}}\right)$$

$$\|x(t)\| \le \left(\left(\left\|\sum_{I=1}^{n} A_i\right\| \|x(t)\| + LC_1(t)\|x(t)\| + \|x(t)\| + \|B\|\|u(\tau)\|\right) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{t^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}\right) + (C_1(t) + B + 1) \left(\|x(s-\tau)\| \frac{1}{\Gamma(1+2\alpha)} \left(\left(\frac{t^\mu - a^\mu}{\mu}\right)^{2\alpha} - \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{2\alpha}\right) + \left[\delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{t^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+2\alpha)} \left(\left(\frac{(t^\mu - \tau^\mu)}{\mu}\right)^{\alpha} - \left(\frac{(t)^\mu - (2\tau)^\mu}{\mu}\right)^{2\alpha}\right) + \cdots + [\delta_i(\tau) E_\alpha \sigma_0(i\tau) (\frac{i\tau^\mu}{\mu})^{2\alpha}] \frac{1}{\Gamma(1+\alpha)} \left(\frac{((t)^\mu - (i\tau)^\mu)}{\mu}\right)^{2\alpha}\right)$$

$$\|x(t)\| \le \left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1\right) {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + \|B\| {}^{RK}_{a}D_t^{-2\alpha,\mu} \|u(\tau)\|\right) + (C_1(t) + B + 1) \left(t\|x(t)\| \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^\mu - a^\mu}{\mu}\right)^{2\alpha} + \left[\delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{\tau^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+2\alpha)} \left(\left(\frac{\tau^\mu}{\mu}\right)^{2\alpha}\right) + \cdots + \left[\delta_i(\tau) E_\alpha \sigma_0(i\tau) \left(\frac{i\tau^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha}\right)$$

$$\|x(t)\| \le \left(\left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1\right) + t(C_1(t) + B + 1)\right) {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + \|B\| {}^{RK}_{a}D_t^{-2\alpha,\mu} \|u(\tau)\|\right) + (C_1(t) + B + 1) \left(\left[\delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{\tau^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+2\alpha)} \left(\left(\frac{\tau^\mu}{\mu}\right)^{2\alpha}\right) + \cdots + \left[\delta_i(\tau) E_\alpha \sigma_0(i\tau) \left(\frac{i\tau^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha}\right)$$

$$\|x(t)\| \le \left(\left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1\right) + t(C_1(t) + B + 1)\right) {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + \|B\| {}^{RK}_{a}D_t^{-2\alpha,\mu} \|u(\tau)\|\right) + (C_1(t) + B + 1) \left([\delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{\tau^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}] \frac{1}{\Gamma(1+2\alpha)} \left(\left(\frac{\tau^\mu}{\mu}\right)^{2\alpha}\right) + \cdots + [\delta_i(\tau) E_\alpha \sigma_0(i\tau) (\frac{i\tau^\mu}{\mu})^{2\alpha}] \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha}\right)$$

$$\|x(t)\| \le \left((\|B\|\|u(\tau)\|) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^\mu - a^\mu}{\mu}\right)^{2\alpha}\right) + \left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1\right) + t(C_1(t) + B + 1)\right) {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + (C_1(t) + B + 1)\left(\left[\delta_1(\tau) E_\alpha \sigma_0(\tau) \left(\frac{\tau^\mu}{\mu} - \frac{a^\mu}{\mu}\right)^{2\alpha}\right] \frac{1}{\Gamma(1+\alpha)} \left(\left(\frac{\tau^\mu}{\mu}\right)^{2\alpha}\right) + \cdots + [\delta_i(\tau) E_\alpha \sigma_0(i\tau) (\frac{i\tau^\mu}{\mu})^{2\alpha}] \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha}\right)$$

$$\delta_1(t) = \left((\|B\|\|u(\tau)\|) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^\mu - a^\mu}{\mu}\right)^{2\alpha}\right),$$

$$\sigma_0(t) = \left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + LC_1(t) + 1\right) + t(C_1(t) + B + 1)\right)$$

$\|x(t)\| \leq (\delta_1(t)) + (C_1(t)+B+1)\Big(\delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\frac{\tau^\mu}{\mu}\right)^{2\alpha}]\ \frac{1}{\Gamma(1+2\alpha)}((\frac{\tau^\mu}{\mu})^{2\alpha}) + \cdots +$
$[\delta_i(\tau)E_\alpha\sigma_0(i\tau)(\frac{i\tau^\mu}{\mu})^{2\alpha}]\ \frac{1}{\Gamma(1+\alpha)}\left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha}\Big) + \sigma_0(t)\,{}^{RK}_{a}D_t^{-\alpha,\mu}\|x(t)\|$

$\|x(t)\| \leq \delta_1((i+1)\tau) + (C_1(t)+B+1)\left[\frac{1}{\Gamma(1+2\alpha)}\left(\frac{(\tau)^\mu)}{\mu}\right)^{2\alpha}\left[\sum_{j=1}^{i}\delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\frac{j\tau^\mu}{\mu}\right)^{2\alpha}\right]\right] +$
$\sigma_0((i+1)\tau)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}\|x(t)\|$

Set $\delta_{i+1}(\tau) = \delta_1((i+1)\tau) + (C_1(t)+B+1)\left[\frac{1}{\Gamma(1+2\alpha)}\left(\frac{(\tau)^\mu)}{\mu}\right)^{2\alpha}\left[\sum_{j=1}^{i}\delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\frac{j\tau^\mu}{\mu}\right)^{2\alpha}\right]\right]$

Using lemma (3.10) , we obtain that for $t \in (i\tau,(i+1)\tau]$, implies that

$$\|x(t)\| \leq \delta_{i+1}(\tau)\ E_\alpha\sigma_0((i+1)\tau)\left(\frac{\tau^\mu}{\mu}-\frac{a^\mu}{\mu}\right)^{2\alpha}$$

$$\|x(t)\| \leq \delta_{i+1}(\tau)\ E_\alpha\sigma_0((i+1)\tau)\left(\frac{((i+1)\tau)^\mu}{\mu}\right)^{2\alpha}$$

Finally for $t \in ((n+1)\tau, T], 1 \leq i \leq n$ .Then it follows that

$\|x(t)\| \leq \left((\|\sum_{I=1}^{n}A_i\| + LC_1(t)+1) + t(C_1(t)+B+1) + \|B\|\|u(\tau)\|\right)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha-1}\ \frac{d\tau}{\tau^{1-\mu}} +$
$(C_1(t)+B+1)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha-1}\|x(t-s)\|\ \frac{d\tau}{\tau^{1-\mu}}$

$\|x(t)\| \leq \Big(\big((\|\sum_{I=1}^{n}A_i\| + LC_1(t)+1) + t(C_1(t)+B+1)\big)\|x(t)\| +$
$\|B\|\|u(\tau)\|\Big)\frac{1}{2\alpha\Gamma(2\alpha)}\left(\frac{t^\mu-a^\mu}{\mu}\right)^{2\alpha} + (C_1(t)+B+1)\Big(\frac{1}{\Gamma(2\alpha)}\int_a^\tau\left(\frac{t^\mu-s^\mu}{\mu}\right)^{2\alpha-1}[\|x(s-\tau)\|]\frac{ds}{s^{1-\mu}} +$
$\frac{1}{\Gamma(2\alpha)}\int_\tau^{2\tau}\left(\frac{t^\mu-s^\mu}{\mu}\right)^{2\alpha-1}[\|x(s-\tau)\|]\frac{ds}{s^{1-\mu}} + \cdots + \frac{1}{\Gamma(2\alpha)}\int_{n\tau}^{(n+1)\tau}\left(\frac{t^\mu-s^\mu}{\mu}\right)^{2\alpha-1}[\|x(s-\tau)\|]\frac{ds}{s^{1-\mu}} +$
$\frac{1}{\Gamma(2\alpha)}\int_{(n+1)\tau}^{t}\left(\frac{t^\mu-s^\mu}{\mu}\right)^{2\alpha-1}[\|x(s-\tau)\|]\frac{ds}{s^{1-\mu}}\Big)$

$\|x(t)\| \leq \Bigg(\Big(\big((\|\sum_{I=1}^{n}A_i\| + LC_1(t)+1) + t(C_1(t)+B+1)\big)\|x(t)\| +$
$\|B\|\|u(\tau)\|\Big)\frac{1}{\Gamma(1+2\alpha)}\left(\frac{t^\mu-a^\mu}{\mu}\right)^{2\alpha}\Bigg) + \big((C_1(t)+B+1)\big)\Big(\|x(s-\tau)\|\frac{1}{\Gamma(1+2\alpha)}((\frac{t^\mu-a^\mu}{\mu})^{2\alpha} -$
$\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{2\alpha}) + [\delta_1(\tau)E_\alpha\sigma_0\tau^{2\alpha}]\ \frac{1}{\Gamma(1+2\alpha)}((\frac{(t^\mu-\tau^\mu)}{\mu})^{2\alpha} - (\frac{(t)^\mu-(2\tau)^\mu}{\mu})^{2\alpha}) + \cdots +$
$[\delta_n(\tau)E_\alpha\sigma_0(n\tau^{2\alpha})]\ \frac{1}{\Gamma(1+2\alpha)}((\frac{((t)^\mu-(n\tau)^\mu)}{\mu})^{2\alpha} - (\frac{((t)^\mu-((n+1)n\tau)^\mu)}{\mu})^{2\alpha}) +$
$[\delta_{n+1}(\tau)E_\alpha\sigma_0(\frac{(n+1)\tau}{\mu})^{2\alpha}]\ \frac{1}{\Gamma(1+2\alpha)}\left(\frac{((t)^\mu-((n+1)\tau)^\mu)}{\mu}\right)^{2\alpha}\Big)$

$\|x(t)\| \leq \Big((\|\sum_{I=1}^{n}A_i\| + LC_1(t)+1) + t(C_1(t)+B+1)\,{}^{RK}_{a}D_t^{-2\alpha,\mu}\|x(t)\| +$
$\|B\|\,{}^{RK}_{a}D_t^{-2\alpha,\mu}\|u(\tau)\|\Big) + \big((C_1(t)+B+1)\big)\Big(\left\|\int_0^t x(s)ds\right\|\frac{1}{\Gamma(1+\alpha)}\left(\frac{\tau^\mu-a^\mu}{\mu}\right)^{2\alpha} +$
$[\delta_1(\tau)E_\alpha L\tau^{2\alpha}]\ \frac{1}{\Gamma(1+2\alpha)}((\frac{(\tau^\mu)}{\mu})^{2\alpha}) + \cdots + [[\delta_n(\tau)E_\alpha\sigma_0(n\tau)^{2\alpha}]\ \frac{1}{\Gamma(1+2\alpha)}\left(\frac{(\tau)^\mu}{\mu}\right)^{2\alpha} +$
$[\delta_{n+1}(\tau)E_\alpha\sigma_0((n+1)\tau^{2\alpha})]\ \frac{1}{\Gamma(1+2\alpha)}\left(\frac{((t)^\mu-((n+1)\tau)^\mu)}{\mu}\right)^{2\alpha}\Big)$

$\|x(t)\| \leq \Big((\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1) + t(C_1(t) + B + 1)\, {}^{RK}_{a}D_t^{-\alpha,\mu} \|x(t)\| + \|B\|\, {}^{RK}_{a}D_t^{-\alpha,\mu} \|u(\tau)\| \Big) + \big((C_1(t) + B + 1)\big)\Big(t\|x(t)\| \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha} + [\delta_1(\tau) E_\alpha L\tau^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} (\left(\frac{(\tau^{\mu})}{\mu}\right)^{2\alpha}) + \cdots + [[\delta_n(\tau) E_\alpha \sigma_0 (n\tau)^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^{\mu})}{\mu}\right)^{2\alpha} + [\delta_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau^{2\alpha})]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{((t)^{\mu} - ((n+1)\tau)^{\mu})}{\mu}\right)^{2\alpha} \Big)$

$\|x(t)\| \leq \Big((\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1) + t(C_1(t) + B + 1)\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + \|B\|\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|u(\tau)\| \Big) + \big((C_1(t) + B + 1)\big)\Big(t\|x(t)\| \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha} + [\delta_1(\tau) E_\alpha L\tau^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} (\left(\frac{(\tau^{\mu})}{\mu}\right)^{2\alpha}) + \cdots + [[\delta_n(\tau) E_\alpha \sigma_0 (n\tau)^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^{\mu})}{\mu}\right)^{2\alpha} + [\delta_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau^{2\alpha})]\, \frac{1}{\Gamma(1+\alpha)} \left(\frac{((t)^{\mu} - ((n+1)\tau)^{\mu})}{\mu}\right)^{2\alpha} \Big)$

$\|x(t)\| \leq \Big((\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1) + t(C_1(t) + B + 1)\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| + \|B\|\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|u(\tau)\| \Big) + \big((C_1(t) + B + 1)\big)\Big([\delta_1(\tau) E_\alpha L\tau^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} (\left(\frac{(\tau^{\mu})}{\mu}\right)^{2\alpha}) + \cdots + [[\delta_n(\tau) E_\alpha \sigma_0 (n\tau)^{2\alpha}]\, \frac{1}{\Gamma(1+\alpha)} \left(\frac{(\tau)^{\mu}}{\mu}\right)^{2\alpha} + [\delta_{n+1}(\tau) E_\alpha \sigma_0((n+1)\tau^{2\alpha})]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{((t)^{\mu} - ((n+1)\tau)^{\mu})}{\mu}\right)^{2\alpha} \Big)$

$\|x(t)\| \leq (\|B\|\|u(\tau)\|)) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha} + \Big((\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1) + t(C_1(t) + B + 1)\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\| \Big) + \big((C_1(t) + B + 1)\big)\Bigg([\delta_1(\tau) E_\alpha \sigma_0 \left(\frac{\tau^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha}]\, \frac{1}{\Gamma(1+\alpha)} (\left(\frac{(\tau^{\mu})}{\mu}\right)^{2\alpha}) + \cdots + [[\delta_n(\tau) E_\alpha \sigma_0 \left(\left(\frac{(n\tau)^{\mu}}{\mu} - \frac{(na)^{\mu}}{\mu}\right)\right)^{2\alpha}]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{(\tau)^{\mu}}{\mu}\right)^{2\alpha} + [\delta_{n+1}(\tau) E_\alpha \sigma_0 \left(\frac{((n+1)\tau)^{\mu}}{\mu} - \frac{((n+1)a)^{\mu}}{\mu}\right)^{2\alpha})]\, \frac{1}{\Gamma(1+2\alpha)} \left(\frac{((t)^{\mu} - ((n+1)\tau)^{\mu})}{\mu}\right)^{2\alpha} \Bigg)$

$\delta_1(t) = (\|B\|\|u(\tau)\|)) \frac{1}{\Gamma(1+2\alpha)} \left(\frac{\tau^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha},$

$\sigma_0(t) = \big((\|\sum_{I=1}^{n} A_i\| + LC_1(t) + 1) + t(C_1(t) + B + 1) \big)$

$\|x(t)\| \leq \delta_1(T) + \big((C_1(t) + B + 1)\big)\left[\frac{1}{\Gamma(1+2\alpha)} \left(\frac{((j\tau)^{\mu})}{\mu}\right)^{2\alpha} \left(\sum_{j=1}^{n} \delta_j(\tau) E_\alpha \sigma_0(j\tau) \left(\frac{(j\tau^{\mu})}{\mu}\right)^{2\alpha}\right) + \frac{1}{\Gamma(1+\alpha)} \left(\frac{(T^{\mu}) - ((n+1)\tau^{\mu})}{\mu}\right)^{2\alpha} \delta_{n+1}(\tau) E_\alpha \sigma_0(n+1) \left(\frac{((n+1)\tau^{\mu})}{\mu}\right)^{2\alpha} \right] + \delta_0(t)\, {}^{RK}_{a}D_t^{-2\alpha,\mu} \|x(t)\|$

$Set\ \delta_T(\tau) = \delta_1(T) + \big((C_1(t) + B + 1)\big)\left[\frac{1}{\Gamma(1+2\alpha)} \left(\frac{(j\tau^{\mu})}{\mu}\right)^{2\alpha} \left(\sum_{j=1}^{n} \delta_j(\tau) E_\alpha \sigma_0(j\tau) \left(\frac{(j\tau^{\mu})}{\mu}\right)^{2\alpha}\right) + \frac{1}{\Gamma(1+\alpha)} \left(\frac{(T^{\mu}) - ((n+1)\tau^{\mu})}{\mu}\right)^{2\alpha} \delta_{n+1}(\tau) E_\alpha \sigma_0(n+1) \left(\frac{((n+1)\tau^{\mu})}{\mu}\right)^{2\alpha} \right].$

By using lemma (3.10)for $t \in ((n+1)\tau, T]$, implies that

$$\|x(t)\| \leq \delta_T(\tau)\, E_\alpha \sigma_0(T) \left(\frac{T^{\mu} - a^{\mu}}{\mu}\right)^{2\alpha}$$

Remark (3.1):

The Composition Caputo −Katugampola fractional order nonlinear differential control nonlocal system (1.1)is finite-time stable if it satisfy the following condition $\delta_T(\tau)\, E_\alpha \sigma_0(T)\left(\frac{T^\mu-a^\mu}{\mu}\right)^{2\alpha} \leq \varepsilon$,where $\delta_T(\tau) = (\|B\|\|u(T)\|)\frac{1}{\Gamma(1+2\alpha)}\left(\frac{T^\mu-a^\mu}{\mu}\right)^{2\alpha}$

4.Illustrative examples

In this section, we provide some examples to illustrate the finite time stability of the Caputo −Katugampola fractional order nonlinear differential control nonlocal system (1.1)..

Example (4.1):

Consider the following the Caputo −Katugampola fractional order nonlinear differential control nonlocal system (1.1),

$$\delta(t) = \|B\|\|u(\tau)\|\frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}$$

$$\widehat{\omega}(t) = \left(\left\|\sum_{I=1}^{n} A_i\right\| + \|B\| + L(2C_1(t) + 2)\right)$$

$$\| x(t) \| \leq \left(\|B\|\|u(\tau)\|\frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}\right) E_\alpha\left(\left(\left\|\sum_{I=1}^{n} A_i\right\| + \|B\| + L(2C_1(t) + 2)\right)\left(\frac{t^\mu - a^\mu}{\mu}\right)^{2\alpha}\right)$$

$${}^{CK}_{a}D^{\alpha,\mu}_{t}\left({}^{CK}_{a}D^{\alpha,\mu}_{t}x(t)\right) = \sum_{I=1}^{n} A_i\begin{pmatrix}\sin(t)\\ cos(t)\end{pmatrix} + \begin{pmatrix} t\sin(t-0.1)\, {}^{RK}_{a}D^{\alpha,\mu}_{t}(\sin(t-0.1)) \\ tcos(t-0.1)\, {}^{RK}_{a}D^{\alpha,\mu}_{t}(cos(t-0.1))\end{pmatrix}\begin{pmatrix}\sin(t)\, {}^{RK}_{a}D^{\alpha,\mu}_{t}(t) \\ \cos(t)\, {}^{RK}_{a}D^{\alpha,\mu}_{t}(t)\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u(t).$$

where $u(t) = [u_1(t), u_2(t)]^T$, is a vector control functions, $A_1 = \begin{pmatrix}1 & -3\\ -2 & 0\end{pmatrix}, A_2 = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}, A_3 = \begin{pmatrix}1 & 1\\ -3 & 0\end{pmatrix}, B = \begin{pmatrix}7 & 1\\ -2 & -4\end{pmatrix}$,We can efficiently verify that $\|A_1\| = \sqrt{11}, \|A_2\| = 2, \|A_3\| = \sqrt{13}, \|B\| = 2\sqrt{14}$, and. Using inequality (3.6) in Theorem (3.1.5) with $\alpha \in (0,1), L = 1, a = 0.1, t = 0.2, \in [-\tau, 0], \|B\| = 1, u(t) = 1$

solution :

$$\| x(t) \| \leq \left(\|B\|\|u(\tau)\|\frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}\right) E_\alpha\left(\left(\|\textstyle\sum_{I=1}^{n} A_i\| + \|B\| + L(2C_1(t) + 2)\right)\left(\frac{t^\mu - a^\mu}{\mu}\right)^{2\alpha}\right)$$

$$\| x(t) \| \leq \left(\frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^\mu - a^\mu)^{2\alpha}\right)\left(\sum_{k=1}^{\infty}\left(\|\textstyle\sum_{i=1}^{n} A_i\| + \|B\| + L\left(\frac{2\mu^{\alpha k}}{\Gamma(1-\alpha k)}\left((t^\mu - a^\mu)\right) + 2\right)\right)\right)\frac{\left(\frac{t^\mu-a^\mu}{\mu}\right)^{2\alpha k}}{\Gamma(\alpha k+1)}$$

$$\| x(t) \| \leq \left(\frac{\mu^{1-2\alpha}}{2\alpha\Gamma(2\alpha)}(t^{\mu}-a^{\mu})^{2\alpha}\right)\left(\left(\left(\left(\sqrt{11}+\ 2+\sqrt{13}\right)+2\sqrt{14}+\sum_{k=1}^{\infty}\left(\left(\frac{2\mu^{\alpha k}}{\Gamma(1-\alpha k)}\left((t^{\mu}-a^{\mu})\right)+2\right)\right)\right)\frac{t^{\alpha k}\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{2\alpha k}}{\Gamma(\alpha k+1)}\right)$$

$$\| x(t) \| \leq \frac{\left(\frac{t^{\mu}}{\mu}-\frac{a^{\mu}}{\mu}\right)^{\alpha}}{\Gamma(1+\alpha)}\left(\sqrt{11}+1+\left(t\left(2+\sqrt{13}+2\sqrt{14}\right)\right)+\sum_{i=0}^{\infty}\left(t\ \frac{\mu^{\alpha i}(t^{\mu}-a^{\mu})\left(\left(\frac{t^{\mu}-a^{\mu}}{\mu}\right)^{\alpha i}\right)}{\Gamma(1-\alpha i)\Gamma(\alpha i+1)}\right)\right)$$

Table1. The value of ε, for $a = 0.1, \alpha = 0.1$

t	**μ = 0.1**	**μ = 0.2**	**μ = 0.3**	**μ = 0.4**	**μ = α**	**μ = 0.6**	**μ = 0.7**	**μ = 0.8**	**μ = 0.9**
0.2	258.5412	272.2187	254.5413	231.3660	209.2953	189.9299	173.3902	159.3732	147.4924
0.4	1.0e+03 *2.5646	1.0e+03 *2.0480	1.0e+03 *1.5440	1.0e+03 *1.1833	1.0e+03 *0.9329	1.0e+03 *0.7564	1.0e+03 *0.6289	1.0e+03 *0.5343	1.0e+03 *0.4624
0.6	1.0e+04 *1.2265	1.0e+04 *0.8253	1.0e+04 *0.5442	1.0e+04 *0.3751	1.0e+04 *0.2718	1.0e+04 *0.2058	1.0e+04 *0.1619	1.0e+04 *0.1314	1.0e+04 *0.1095
0.8	1.0e+04 *4.3460	1.0e+04 *2.6063	1.0e+04 *1.5679	1.0e+04 *1.0046	1.0e+04 * 0.6864	1.0e+04 * 0.4961	1.0e+04 * 0.3756	1.0e+04 *0.2957	1.0e+04 *0.2402
1	1.0e+05 *1.2886	1.0e+05 *0.7148	1.0e+05 *0.4039	1.0e+05 * 0.2462	1.0e+05* 0.1617	1.0e+05 *0.1132	1.0e+05 *0.0836	1.0e+05 *0.0646	1.0e+05 * 0.0517

Table2. The value of ε, for $a = 0.1, \alpha = 0.5$

t	μ=0. 1	μ=0. 2	μ=0. 3	μ=0. 4	μ=α	μ=0. 6	μ=0. 7	μ=0. 8	μ=0. 9
0.2	299.4857	119.2382	66.4523	43.2298	30.8623	23.4652	18.6739	15.3824	13.0157
0.4	1.0e+03 *2.2507	1.0e+03 *0.6676	1.0e+03 *0.2959	1.0e+03 *0.1606	1.0e+03 *0.0992	1.0e+03 *0.0670	1.0e+03 *0.0484	1.0e+03 *0.0368	1.0e+03 *0.0291
0.6	1.0e+04 *1.0146	1.0e+04 *0.2494	1.0e+04 *0.0951	1.0e+04 *0.0458	1.0e+04 *0.0256	1.0e+04 *0.0159	1.0e+04 *0.0107	1.0e+04 *0.0077	1.0e+04 *0.0058
0.8	1.0e+04 *3.5339	1.0e+04 *0.7639	1.0e+04 *0.2624	1.0e+04 *0.1157	1.0e+04 *0.0602	1.0e+04 *0.0353	1.0e+04 *0.0226	1.0e+04 *0.0155	1.0e+04 *0.0113
1	1.0e+04 *3.0689	1.0e+04 *2.0669	1.0e+04 *0.6594	1.0e+04 *0.2735	1.0e+04 *0.1353	1.0e+04 *0.0759	1.0e+04 *0.0469	1.0e+04 *0.0312	1.0e+04 *0.0220

Table3. The value of ε, for $a = 0.1, \alpha = 0.9$

t	**μ = 0.1**	**μ = 0.2**	**μ = 0.3**	**μ = 0.4**	**μ = α**	**μ = 0.6**	**μ = 0.7**	**μ = 0.8**	**μ = 0.9**
0.2	1.0e+03 *1.6292	1.0e+03 * 0.2373	1.0e+03 *0.0758	1.0e+03 *0.0338	1.0e+03 *0.0181	1.0e+03 * 0.0110	1.0e+03 *0.0072	1.0e+03 * 0.0050	1.0e+03 *0.0036
0.4	1.0e+03 *9.9009	1.0e+03 *1.0869	1.0e+03 *0.2808	1.0e+03 *0.1065	1.0e+03 *0.0506	1.0e+03 *0.0278	1.0e+03 *0.0170	1.0e+03 *0.0112	1.0e+03 *0.0078
0.6	1.0e+03 *7.3031	1.0e+03 *3.8031	1.0e+03 *0.8373	1.0e+03 *0.2794	1.0e+03 *0.1196	1.0e+03 *0.0605	1.0e+03 *0.0345	1.0e+03 *0.0215	1.0e+03 *0.0144
0.8	1.0e+03 *6.2617	1.0e+03 *3.2411	1.0e+03 *2.2218	1.0e+03 *0.6726	1.0e+03 *0.2654	1.0e+03 *0.1254	1.0e+03 *0.0675	1.0e+03 *0.0400	1.0e+03 *0.0256

1	958.2335	858.8854	777.8322	611.4592	570.8595	255.7875	131.4636	74.9320	46.2725

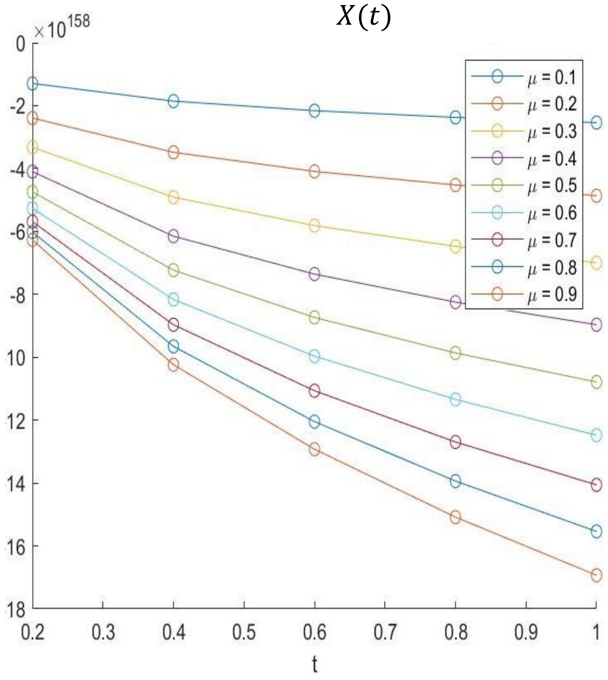

Figure 1.of table1.

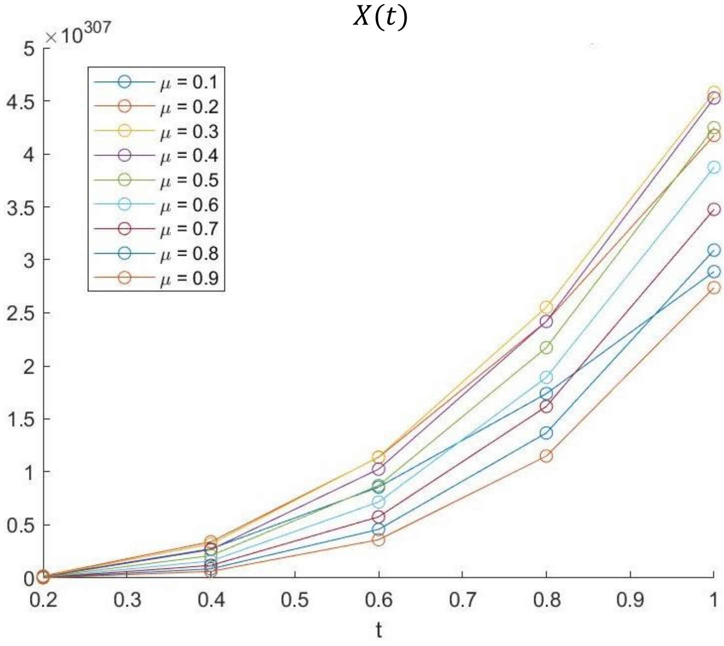

Figure 2.of table2.

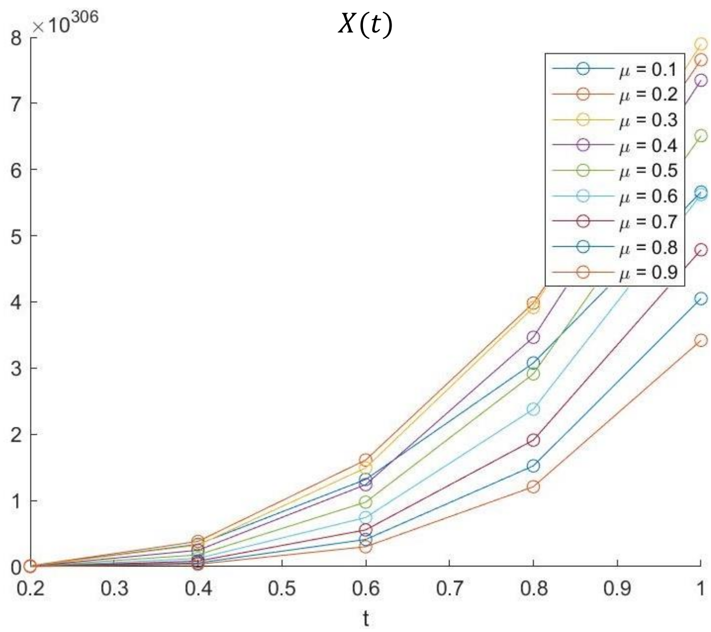

Figure 3.of table3.

5. Conclusion

i. The Composition Caputa and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system 1.1 is very difficult to study since the fractional derivative types of expression are difficult to discuss.

ii . The uniqueness and existence depended on the generalized Gronwall Inequality of The Composition Riemann – Caputa fractional derivative was presented first time and make good role in stability of the system .

iii . The stability of finite time for the given system depend on maximal interval or on step size of maximal interval to obtain the guarantee estimation of epsilon

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