Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 3, No. 1, 2025, Pages 101-116
DOI: 10.47831/mjpas.v3i1.140

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage:
https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

fractional optimal control of systems using Bernoulli wavelets

Murtadaa Muhisin Jasim[1*], Masood Rodaki[2]

[1,2] Shiraz Azad University, College of Applied Mathematics Science, Mathematics Department

[*] Corresponding author E-mail: murtaz94a@gmail.com

Article Info.	**Abstract**
Article history: Received 10 March 2024 Accepted 24 April 2024 Publishing 30 January 2025	In order to provide a new method for partial optimal control of systems, this work uses Bernoulli wavelets and uses Caputo's formula to solve partial optimal control problems (FOCPs) with inequality constraints. For the new optimization technique, the work uses partial order Bernoulli wave functions (F-BWFs) as basic functions. The answer using this method is expressed in terms of F-BWF where the coefficients have not yet been found. To simplify FOCPs in a system of nonlinear algebraic equations the procedure entails transforming inequality constraints into equality requirements by using operational matrices of fractional integration and F-BWFs and using the multipliers technique Lagrange. Numerical examples confirm the validity of the proposed strategy and demonstrate its correctness and efficiency when compared to the analytical or approximate answers provided by other methods.

The official journal published by the College of Education at Mustansiriya University

***Keywords**:* Partially ideal control - Inequality constraints - Caputo definition - Bernoulli wavelet functions - Fractional-order Bernoulli wavelet functions.

1. Introduction

Within the modeling of a wide range of phenomena in science and engineering, many researchers have demonstrated the utility of (FDEs). [1],[2],[3],[4]. After reviewing these studies, it was found that FDEs are considered successful in describing a wide range of complex physical and engineering challenges. [5]

Due to the lack of analytic solutions for most (FDEs), the utilization of approximate and numerical techniques becomes essential. Various analytical and numerical methods have been proposed to solve FDEs, including the extrapolation method [6], [7], [8], [9], [10], [11], [12], [13], [14]

A branch of optimization theory called optimal control theory applies to a variety of fields including industry, engineering, and science. Its main goal is to minimize expenses or maximize rewards. The field of optimal control issues has been the subject of much research [15], [16], [17], [18], [19], [20] Swann 1990 [21]was studied in detail.

By researching new areas in mathematics, we find that partial optimal control theory is the best. Although it is constantly being developed, it is possible to use its various definitions for fractional derivatives. For example, for fractional optimal control problems (FOCPs), we use fractional Riemann-Liouville and Caputo derivatives [22], Clustering techniques and Legendre's multiwavelet rules have been effectively applied by researchers such as Yousefi et al. (2011) [23] To estimate the solutions of FOCPs, a method for multi-dimensional FOCPs based on Bernstein OM polynomials is presented [24] ,research has been done on using OM polynomials and the Ritz technique to solve FOCPs [25], [26], neural network applications have been studied [27] in order to solve a class of FOCPs.

An analytical and numerical method for solving unconstrained distributed-order partial optimal control problems (FOCPs) was presented[27], the approximation of solutions for FOCPs using Bernoulli polynomials was investigated, [28], [29] presented a numerical method using wavelet LCs for binary FOCPs. dimensions, and using generalized partial order Bernoulli-Legendre functions, two-dimensional FOCPs were solved in a follow-up study [30].

Researchers in [31], [32], [33], [34] are among those whose works interested readers might consult for further examination of FOCPs.

This paper's primary goal is to suggest an optimization strategy for the following FOCPs min based on the fractional-order Bernoulli wavelet functions (F-BWFs)

$$\boldsymbol{J} = \int_0^1 F\big(t, x(t), u(t)\big)dt \tag{1}$$

susceptible to limitations on inequality and the fractional dynamical system

$$\begin{aligned} &{}_0^C D_t^{\nu} x(t) = \mathcal{G}\left(t, x(t), u(t)\right), \qquad 0 < \nu, \;\; t \in [0,1], \\ &S_j(t, x(t), u(t)) \le 0, \qquad\qquad j = 1, \ldots, s, \end{aligned} \tag{2}$$

and the initial conditions

$$x(0) = x_0, \dot{x}(0) = x_1, \ldots, x^{[\nu]}(0) = x_{[\nu]}, \tag{3}$$

The partial optimal control of systems using Bernoulli wavelets is then addressed by introducing an optimization technique based on F-BWF. A new OM and OM product of fractional integration and a numerical method for solving equations (1)–(3) are established. The structure of this document is as follows: In Section 2, the basic concepts of fractional calculus and fractional ordering of Bernoulli wavelet functions were addressed, in Section 3 the F-BWF operational matrices were developed, in Section 4 a digital technique based on F-BWF was presented, and in Section 5 illustrative examples were provided. It demonstrates the accuracy and effectiveness of the current approach, and finally Section 6 provides a summary of the main results.

2. Definitions and Mathematical Preliminaries:

This part goes over the F-BWFs and introduces fractional calculus.

Definition 1: (Hassani et al. 2019a, b) defines the Caputo fractional derivative of order v of f(t) for q - 1< v < q.

$${}_0^C D_t^{\nu} f(t) = \begin{cases} \dfrac{1}{\Gamma(q-\nu)} \displaystyle\int_0^t \frac{f^{(q)}(s)}{(t-s)^{(\nu+1-q)}} ds, & q-1 < \nu < q, \;\; q \in \mathbb{N}, \\ \dfrac{d^q f(t)}{dt^q}, & \nu = q, \end{cases} \tag{4}$$

where T(.) represents the gamma function defined for z > 0 and q = [v]+1 indicates the integer component of v

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \mathrm{d}t.$$

Definition 2: [30] defines the Riemann-Liouville fractional integral operator of order V of f(t)

$$I_t^{\nu} f(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t \frac{f(s)}{(t-s)^{1-\nu}} \,\mathrm{d}s, & \nu > 0, t > 0, \\ f(t), & \nu = 0. \end{cases} \tag{5}$$

The following statement provides the useful relationship between the Caputo operator and the Riemann-Liouville operator [30]

$$I_t^{\nu}\,{}_0^C D_t^{\nu} f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0)\frac{t^i}{i!}, \qquad t>0, \;\; n-1<\nu\le n \tag{6}$$

It is also important to note the following useful property: [35]

$$ {}_0^C D_t^{\nu} t^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m-\nu+1)} t^{m-\nu}, & m = q, q+1, \dots, \\ 0, & m = 0, 1, \dots, q-1 \end{cases} \tag{7}$$

where q -1 < v < q.

2.1 Segment-Based Bernoulli Wavelets:

On the interval [0,1) specified, Partial ordering of Bernoulli wavelets of order a, Ψa n,m; n=1; 2;...; 2k~-1, m = 0; 1;...; M

$$\psi_{n,m}^{\alpha}(t) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{\beta}_m(2^{k-1}t^{\alpha}-n+1), & \dfrac{n-1}{2^{k-1}} \le t^{\alpha} < \dfrac{n}{2^{k-1}} \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

that $ß^{\sim}(2^{k-1}t^{a} - n + 1)$ m=0 , m>0

$$= \begin{cases} 1, \\ \dfrac{1}{\sqrt{\dfrac{(-1)^{(m-1)}(m!)^2}{(2m)!}\beta_{2m}}}\beta_m(2^{k-1}t^{\alpha}-n+1) \end{cases} \tag{9}$$

in which Bernoulli polynomials of rank m on [0, 1] are denoted by $ß_m(t)$. The definition of $ß_m(t)$, the degree m Bernoulli polynomials, is given by (Keshavarz et al. 2015).

$$\beta_m(t) = \sum_{i=1}^{m} \binom{m}{i} \beta_{m-i} t^i \tag{10}$$

where ßi are rational numbers, often known as Bernoulli numbers, that can be found by applying the trigonometric function series expansion

$$\frac{t}{e^t-1} = \sum_{i=0}^{\infty} \beta_i \frac{t^i}{i!}$$

The first few Bernoulli numbers are $\beta_0 = 1, \beta_1 = \dfrac{-1}{2}, \beta_2 = \dfrac{1}{6}, \beta_4 = \dfrac{-1}{30}$

$$\beta_0(t) = 1, \beta_1(t) = t - \frac{1}{2}, \beta_2(t) = t^2 - t + \frac{1}{6}$$
$$\beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

2.2 The Functions Approximation:

Any square-integrable stochastic function f(t) can be expanded in the interval [0, 1] using F-BWFs such as (Rahimkhani et al. 2016).

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}^{\alpha}(t) \tag{11}$$

Equation (11) reduces the infinite series so that f(t) is roughly expressed in terms of the F-BWFs

$$f(t)\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M}c_{n,m}\psi_{n,m}^{\alpha}(t)=C^{\mathrm{T}}\Psi^{\alpha}(t) \tag{12}$$

Where the transposed T and supplied by are the unknown vector C and Ψa(t) which are two perpendicular vectors k-1(M + 1)

$C = [c1;0; c1;1; . . .; c1;M; c2;0; c2;1; . . .; c2;M; . . .; c2^{k-1};0; c2^{k-1};1; . . .; c2^{k-1};M]^T$

$\Psi^a(t)= [\ \Psi_1^a{,}_0 (t),\ \Psi_1^a{,}_1 (t), \ldots, \Psi_1^a, M(t),\ \Psi_2^a{,}_0 (t),\ \Psi_2^a{,}_1 (t), \ldots, \Psi_2^a, M(t),\ldots, \Psi_{2k-1}^a{,}_0(t),\ \Psi_{2k-1}^a{,}_1(t), \ldots, \Psi_{2k-1}^a,\ M(t)]^T,$

And $C^T = F^TD^{-1}$, (13)

$D= \{ \Psi^a, \Psi^a\} = \int_0^1 \Psi^a(t)\ \Psi^{At}(t)\ t^{a-1}\ dt,$

$$\begin{aligned}
&D=\left[d_{n,m,i,j}\right],\quad d_{n,m,i,j}=\langle\psi_{n,m}^{\alpha},\psi_{i,j}^{\alpha}\rangle,\\
&F=\left[f_{1,0},f_{1,1},\ldots,f_{1,M},f_{2,0},f_{2,1},\ldots,f_{2,M},\ldots,\right.\\
&\quad\left.f_{2^{k-1},0},f_{2^{k-1},1},\ldots,f_{2^{k-1},M}\right]^{\mathrm{T}},\\
&f_{ij}=\langle f,\psi_{ij}^{\alpha}\rangle=\int_0^1 f(t)\psi_{ij}^{\alpha}(t)t^{\alpha-1}\mathrm{d}t,\\
&\quad i=1,2,\ldots,2^{k-1},\ j=1,2,\ldots,M,
\end{aligned} \tag{14}$$

3. The Matrices of Operation:

Determining the F-BWFs OM of the partial integration and OM of the product is the primary goal of this section.

3.1 The Fractional Integration Operational Matrix:

The fractional integral of Riemann-Liouville for the vector Ψa(t), as stated in Equation (13) can be acquired as

$$I_t^{\nu}\Psi^{\alpha}(t)\simeq P^{(\nu,\alpha)}\Psi^{\alpha}(t) \tag{15}$$

where P(v,a) is the Riemann–Liouville fractional integration described by the

$2^{k-1}(M + 1) * 2^{k-1}(M + 1)$ dimensional OM M

$$
I_t^{v}\Psi^{\alpha}(t)=\begin{bmatrix} I_t^{v}\psi_{1,0}^{\alpha}(t)\\ I_t^{v}\psi_{1,1}^{\alpha}(t)\\ \vdots\\ I_t^{v}\psi_{1,M}^{\alpha}(t)\\ I_t^{v}\psi_{2,0}^{\alpha}(t)\\ I_t^{v}\psi_{2,1}^{\alpha}(t)\\ \vdots\\ I_t^{v}\psi_{2,M}^{\alpha}(t)\\ \vdots\\ I_t^{v}\psi_{2^{k-1},0}^{\alpha}(t)\\ I_t^{v}\psi_{2^{k-1},1}^{\alpha}(t)\\ \vdots\\ I_t^{v}\psi_{2^{k-1},M}^{\alpha}(t) \end{bmatrix} \simeq \begin{bmatrix} \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{1,0}\psi_{n,m}^{\alpha}(t)\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{1,1}\psi_{n,m}^{\alpha}(t)\\ \vdots\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{1,M}\psi_{n,m}^{\alpha}(t)\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2,0}\psi_{n,m}^{\alpha}(t)\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2,1}\psi_{n,m}^{\alpha}(t)\\ \vdots\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2,M}\psi_{n,m}^{\alpha}(t)\\ \vdots\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2^{k-1},0}\psi_{n,m}^{\alpha}(t)\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2^{k-1},1}\psi_{n,m}^{\alpha}(t)\\ \vdots\\ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} E_{n,m}^{2^{k-1},M}\psi_{n,m}^{\alpha}(t) \end{bmatrix} = \begin{bmatrix} (E^{1,0})^{T}\Psi^{\alpha}(t)\\ (E^{1,1})^{T}\Psi^{\alpha}(t)\\ \vdots\\ (E^{1,M})^{T}\Psi^{\alpha}(t)\\ (E^{2,0})^{T}\Psi^{\alpha}(t)\\ (E^{2,1})^{T}\Psi^{\alpha}(t)\\ \vdots\\ (E^{2,M})^{T}\Psi^{\alpha}(t)\\ \vdots\\ (E^{2^{k-1},0})^{T}\Psi^{\alpha}(t)\\ (E^{2^{k-1},1})^{T}\Psi^{\alpha}(t)\\ \vdots\\ (E^{2^{k-1},M})^{T}\Psi^{\alpha}(t) \end{bmatrix}
$$

$$
=\begin{bmatrix}
E_{1,0}^{1,0} & E_{1,1}^{1,0} & \cdots & E_{1,M}^{1,0} & \cdots & E_{2^{k-1},0}^{1,0} & E_{2^{k-1},1}^{1,0} & \cdots & E_{2^{k-1},M}^{1,0}\\
E_{1,0}^{1,1} & E_{1,1}^{1,1} & \cdots & E_{1,M}^{1,1} & \cdots & E_{2^{k-1},0}^{1,1} & E_{2^{k-1},1}^{1,1} & \cdots & E_{2^{k-1},M}^{1,1}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
E_{1,0}^{1,M} & E_{1,1}^{1,M} & \cdots & E_{1,M}^{1,M} & \cdots & E_{2^{k-1},0}^{1,M} & E_{2^{k-1},1}^{1,M} & \cdots & E_{2^{k-1},M}^{1,M}\\
E_{1,0}^{2,0} & E_{1,1}^{2,0} & \cdots & E_{1,M}^{2,0} & \cdots & E_{2^{k-1},0}^{2,0} & E_{2^{k-1},1}^{2,0} & \cdots & E_{2^{k-1},M}^{2,0}\\
E_{1,0}^{2,1} & E_{1,1}^{2,1} & \cdots & E_{1,M}^{2,1} & \cdots & E_{2^{k-1},0}^{2,1} & E_{2^{k-1},1}^{1,0} & \cdots & E_{2^{k-1},M}^{1,0}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
E_{1,0}^{2,M} & E_{1,1}^{2,M} & \cdots & E_{1,M}^{2,M} & \cdots & E_{2^{k-1},0}^{2,M} & E_{2^{k-1},1}^{2,M} & \cdots & E_{2^{k-1},M}^{2,M}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
E_{1,0}^{2^{k-1},0} & E_{1,1}^{2^{k-1},0} & \cdots & E_{1,M}^{2^{k-1},0} & \cdots & E_{2^{k-1},0}^{2^{k-1},0} & E_{2^{k-1},1}^{2^{k-1},0} & \cdots & E_{2^{k-1},M}^{2^{k-1},0}\\
E_{1,0}^{2^{k-1},1} & E_{1,1}^{2^{k-1},1} & \cdots & E_{1,M}^{2^{k-1},1} & \cdots & E_{2^{k-1},0}^{2^{k-1},1} & E_{2^{k-1},1}^{2^{k-1},1} & \cdots & E_{2^{k-1},M}^{2^{k-1},1}\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
E_{1,0}^{2^{k-1},M} & E_{1,1}^{2^{k-1},M} & \cdots & E_{1,M}^{2^{k-1},M} & \cdots & E_{2^{k-1},0}^{2^{k-1},M} & E_{2^{k-1},1}^{2^{k-1},M} & \cdots & E_{2^{k-1},M}^{2^{k-1},M}
\end{bmatrix}\Psi^{\alpha}(t)=P^{(v,\alpha)}\Psi^{\alpha}(t) \qquad (16)
$$

Where

$$E^{i,j} = \hat{E}^{i,j} D^{-1},$$
$$\hat{E}^{i,j} = \Big[\hat{E}^{i,j}_{1,0}, \hat{E}^{i,j}_{1,1}, \ldots, \hat{E}^{i,j}_{1,M}, \ldots, \hat{E}^{i,j}_{2,0}, \hat{E}^{i,j}_{2,1}, \ldots, \hat{E}^{i,j}_{2,M}, \ldots,$$
$$\hat{E}^{i,j}_{2^{k-1},0}, \hat{E}^{i,j}_{2^{k-1},1}, \ldots, \hat{E}^{i,j}_{2^{k-1},M}\Big]^{\mathrm{T}},$$
$$\hat{E}^{i,j}_{n,m} = \Big\langle I_t^{\nu} \psi^{\alpha}_{i,j}(t), \psi^{\alpha}_{n,m}(t)\Big\rangle,$$
$$n = 1, \ldots, 2^{k-1}, \quad m = 0, \ldots, M.$$

We select (v = 1, a = 3/2, k = 1, M = 2) to demonstrate the computation process. So, we've

$$P^{(1,\frac{3}{2})} = \begin{bmatrix} \frac{3}{5} & \frac{3\sqrt{3}}{20} & -\frac{3\sqrt{5}}{220} \\ -\frac{3\sqrt{3}}{10} & -\frac{9}{220} & \frac{51\sqrt{15}}{1540} \\ \frac{6\sqrt{5}}{55} & -\frac{39\sqrt{15}}{1540} & -\frac{87}{5236} \end{bmatrix}$$

also by choosing (v = a = 3/2, k = 1, M = 2), we have

$$P^{(\frac{3}{2},\frac{3}{2})} = \begin{bmatrix} 0.3761263890318 & 0.2171566719569 & 0 \\ -0.3956383388839 & -0.1545696576686 & 0.05720570205399 \\ 0.234689686294 & 0.02705211835561 & -0.04961323923435 \end{bmatrix}$$

3.1 The F-BWFs Product Operational Matrix:

$$\Psi^{\alpha}(t)\Psi^{\alpha\mathrm{T}}(t)A \simeq \tilde{A}\Psi^{\alpha}(t) \tag{17}$$

where A˜ is a (M + 1) * (M +1) matrix and A is an arbitrary (M + 1) *1 vector. We can use Ψa(t)ΨaT(t)A to approximate Ψa(t) in order to obtain A˜.

$$\Psi^{\alpha}(t)\Psi^{\alpha\mathrm{T}}(t)A = [a_0(t), \ldots, a_M(t)]^{\mathrm{T}} \tag{18}$$

Where
$$a_i(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} \tilde{a}_{i,n,m}\psi^{\alpha}_{n,m}(t) = \tilde{A}_i^{\mathrm{T}}\Psi^{\alpha}(t) \tag{19}$$

And
$$\tilde{A}_i = [\tilde{a}_{i,1,0}, \tilde{a}_{i,1,1}, \ldots, \tilde{a}_{i,1,M}, \tilde{a}_{i,2,0}, \tilde{a}_{i,2,1}, \ldots, \tilde{a}_{i,2,M}, \ldots, \tilde{a}_{i,2^{k-1},0}, \tilde{a}_{i,2^{k-1},1}, \ldots, \tilde{a}_{i,2^{k-1},M}]^{\mathrm{T}}. \tag{20}$$

Using Eq. (19) we obtain
$$a_i^{k,j} = \Big\langle \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} \tilde{a}_{i,n,m}\psi^{\alpha}_{n,m}(t), \psi^{\alpha}_{k,j}(t)\Big\rangle = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} \tilde{a}_{i,n,m} d_{n,m,k,j}, \quad k = 1, \ldots, 2^{k-1}, \; j = 0, 1, \ldots, M, \tag{21}$$

where

$$a_i^{k,j} = \langle a_i, \psi^{\alpha}_{k,j}\rangle, d_{n,m,k,j} = \langle \psi^{\alpha}_{n,m}, \psi^{\alpha}_{k,j}\rangle. \tag{22}$$

So by considering

$$A_i = \left[a_i^{1,0}, a_i^{1,1}, \ldots, a_i^{1,M}, a_i^{2,0}, a_i^{2,1}, \ldots\ a_i^{2,M}, \ldots, a_i^{2^{k-1},0}, a_i^{2^{k-1},1}, \ldots, a_i^{2^{k-1},M}\right]^{\mathrm{T}},$$

we have

$$A_i^T = \tilde{A}_i^T D, \quad \text{or} \quad \tilde{A}_i^T = A_i^T D^{-1}$$

so that the OM of multiplication can be determined. To demonstrate the computation process, we select (k = 1, M = 2, a = 1) thus, we have

$$A = [A_1, A_2, A_3],\ \tilde{A} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_2 & A_1 + \frac{2\sqrt{5}}{5}A_3 & \frac{2\sqrt{5}}{5}A_2 \\ A_3 & \frac{2\sqrt{5}}{5}A_2 & A_1 + \frac{2\sqrt{5}}{7}A_3 \end{bmatrix}. \tag{23}$$

$$f_{i,j} = \int_0^1 \frac{A}{2^{\frac{k-1}{2}}\alpha} f\left(\left(\frac{x+i-1}{2^{k-1}}\right)^{\frac{1}{\alpha}}\right)\beta_j(x)\mathrm{d}x.$$

3.2 Analysis of Convergence:

Theorems that follow will be helpful in the findings that follow. In this case, we presume

Theorem 1 Suppose f Є $L^2[0\ ,\ 1]$ be a continuous function and $|f(t)| \le M_1\ \forall t \in [0,1]$

And $f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}^{\alpha}(t).$ $D^{-1} = [d_{n,m}^{i,j}]$, $\max|d_{n,m}^{i,j}| = M_2$ then, we have

$$|c_{n,m}| < \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{M} \frac{16j!M_1M_2A}{(2\pi)^{j+1}2^{\frac{k-1}{2}}\alpha}, \tag{24}$$

$$A = \frac{1}{\sqrt{\frac{(-1)^{j-1}(j!)^2}{(2j)!}\beta_{2j}}}$$

Where

Evidence Assume that, using FBWFs, f ~*(t) is an approximation of f(t). Next, we have using Eq. (12)

$$f^*(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}^{\alpha}(t) = C^T\Psi^{\alpha}(t)$$ where $C^T = F^T D^{-1}$. Then, we get

$$c_{n,m} = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{M} f_{i,j} d_{n,m}^{i,j},\ f_{i,j} = \langle f, \psi_{i,j}^{\alpha}\rangle$$ and

$$f_{i,j} = \int_0^1 f(t)\psi_{i,j}^{\alpha}(t)t^{\alpha-1}\mathrm{d}t$$
$$= \int_{(\frac{i-1}{2^{k-1}})^{\frac{1}{\alpha}}}^{(\frac{i}{2^{k-1}})^{\frac{1}{\alpha}}} 2^{\frac{k-1}{2}}Af(t)B_j(2^{k-1}t^{\alpha} - i + 1)t^{\alpha-1}\mathrm{d}t.$$

Here, by changing the variable $2^{k-1} t^{a} - i + 1 = x$,

We know that i$|f(x)| \le M_1, \forall x \in [0,1]$ then, $\left|f(x^{\frac{1}{\alpha}})\right| \le M_1$

By making use of the following Bernoulli polynomial property (Sahu and Saha Ray 2017)

$$\int_0^1 |\beta_j(t)|\mathrm{d}t < \frac{16j!}{(2\pi)^{j+1}}, j \ge 0.$$

we have

$$|f_{i,j}| \le \frac{A}{2^{\frac{k-1}{2}}\alpha}\int_0^1 \left|f\left(\left(\frac{x+i-1}{2^{k-1}}\right)^{\frac{1}{\alpha}}\right)\right| |\beta_j(x)|\mathrm{d}x$$
$$< \frac{16j!M_1A}{(2\pi)^{j+1}2^{\frac{k-1}{2}}\alpha}.$$

On the other hand, $\max |d_{n,m}^{i,j}| = M_2$, so $|d_{n,m}^{i,j}| \le M_2$. Taking into account the conversation above, we obtain the following relation

$$|c_{n,m}| < |\sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M} f_{i,j}d_{n,m}^{i,j}| \le \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M}$$
$$|f_{i,j}||d_{n,m}^{i,j}| < \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M} |f_{i,j}|M_2$$
$$< \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M} \frac{16j!M_1M_2A}{(2\pi)^{j+1}2^{\frac{k-1}{2}}\alpha}.$$

Theorem 2 Assume that f Є $L^2[0, 1]$ be a continuous function and

$|f(t)| \le M_1, \forall t \in [0,1]$. *If* $f^*(t)$ The truncated error criterion E(t) can be determined if the expansion of F-BWFs is truncated as

$$\|E(t)\|_2 = \|f(t) - f^*(t)\|_2$$
$$\le \sum_{m=0}^{M}\sum_{n=2^{k-1}+1}^{\infty} B_{i,j} + \sum_{m=M+1}^{\infty}\sum_{n=1}^{\infty} B_{i,j}, \tag{25}$$

Where

$$B_{i,j} = \sum_{j=0}^{M}\sum_{i=1}^{2^{k-1}} \frac{16j!M_1M_2A}{(2\pi)^{j+1}2^{\frac{k-1}{2}}\alpha^2}, \quad A = \frac{1}{\sqrt{\frac{(-1)^{j-1}(j!)^2}{(2j)!}\beta_{2j}}}$$

Use of Equation as Proof To calculate the truncated error term, let f *(t) represent the truncated F-BWFs expansion

$$E(t) = f(t) - f^*(t) = \sum_{m=0}^{M}\sum_{n=2^{k-1}+1}^{\infty} c_{n,m}\psi_{n,m}^{\alpha}(t)$$
$$+ \sum_{m=M+1}^{\infty}\sum_{n=1}^{\infty} c_{n,m}\psi_{n,m}^{\alpha}(t)$$

then, we derive

$$\|E(t)\|_2 \le \|f(t)-f^*(t)\|_2 \le \left\| \sum_{m=0}^{M} \sum_{n=2^{k-1}+1}^{\infty} c_{n,m}\psi^{\alpha}_{n,m}(t) \right\|_2 + \left\| \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} c_{n,m}\psi^{\alpha}_{n,m}(t) \right\|_2$$

$$\le \sum_{m=0}^{M} \sum_{n=2^{k-1}+1}^{\infty} \left\| c_{n,m}\psi^{\alpha}_{n,m}(t) \right\|_2 + \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} \left\| c_{n,m}\psi^{\alpha}_{n,m}(t) \right\|_2$$

$$= \sum_{m=0}^{M} \sum_{n=2^{k-1}+1}^{\infty} |c_{n,m}| \left\| \psi^{\alpha}_{n,m}(t) \right\|_2 + \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} |c_{n,m}| \left\| \psi^{\alpha}_{n,m}(t) \right\|_2.$$

Additionally, we have

$$\|\psi^{\alpha}_{n,m}(t)\|_2 = \int_0^1 (\psi^{\alpha}_{n,m}(t))^2 t^{\alpha-1} \mathrm{d}t$$

$$= \int_{(\frac{n-1}{2^{k-1}})^{\frac{1}{\alpha}}}^{(\frac{n}{2^{k-1}})^{\frac{1}{\alpha}}} (2^{\frac{k-1}{2}} A\beta_m(2^{k-1}t^{\alpha} - n + 1))^2 t^{\alpha-1} \mathrm{d}t.$$

By changing the variable $2^{k-1} t^a - n + 1 = x$, we get

$$\|\psi^{\alpha}_{n,m}(t)\|_2 = \int_0^1 2^{k-1} A^2 \beta_m(x)^2 \frac{1}{2^{k-1}\alpha} \mathrm{d}x = \frac{A^2}{\alpha} \int_0^1 \beta_m(x)^2 \mathrm{d}x$$

$$= A^2 \frac{1}{\alpha} (-1)^{m-1} \frac{(m!)^2}{(2m)!} \beta_{2m} = \frac{1}{\alpha}.$$

Therefore, with Theorem 1 and the equation above, we obtain

$$\|E(t)\|_2 \le \sum_{m=0}^{M} \sum_{n=2^{k-1}+1}^{\infty} \frac{1}{\alpha} |c_{n,m}| + \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\alpha} |c_{n,m}|$$

$$\le \sum_{m=0}^{M} \sum_{n=2^{k-1}+1}^{\infty} B_{i,j} + \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} B_{i,j},$$

Where $B_{i,j} = \sum_{j=0}^{M} \sum_{i=1}^{2^{k-1}} \frac{16 j! M_1 M_2 A}{(2\pi)^{j+1} 2^{\frac{k-1}{2}} \alpha^2}$

4. An explanation of the suggested approach:

In this part the following FOCPs with inequality constraints are solved using expansion of F-BWFs and their OM

$$\begin{aligned}
&\min \mathcal{J} = \int_0^1 \mathcal{F}(t, x(t), u(t))\, dt, \\
&{}_0^C D_t^\nu x(t) = \mathcal{G}(t, x(t), u(t)), \quad \nu > 0, \ t \in [0,1], \\
&S_j(t, x(t), u(t)) \le 0, \quad j = 1, \dots, s, \\
&x(0) = x_0, x'(0) = x_1, \dots, x^{[\nu]}(0) = x_{[\nu]},
\end{aligned} \tag{26}$$

where the state and control functions are denoted by x(t) and u(t).

To do this, we multiply ${}_0^C D_t^\nu x(t)$ F-BWFs as

$${}_0^C D_t^\nu x(t) \simeq C^T \Psi^\alpha(t). \tag{27}$$

By considering Eq. (15) and the initial conditions given in Eq. (26), we may derive by integrating from order m on both sides of Eq. (27) with respect to t

$$\begin{aligned}
x(t) &= I_t^\nu (C^T \Psi^\alpha(t)) + \sum_{i=0}^{[\nu]} \frac{x_i t^i}{i!} \\
&= C^T P^{(\nu,\alpha)} \Psi^\alpha(t) + \sum_{i=0}^{[\nu]} \frac{x_i t^i}{i!} \\
&= C^T P^{(\nu,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t)
\end{aligned} \tag{28}$$

Where $\sum_{i=0}^{[\nu]} \frac{x_i t^i}{i!} \simeq d^T \Psi^\alpha(t),$

and we suppose $u(t) \simeq U^T \Psi^\alpha(t)$

Our problem is changed to the following problem by replacing Eq. (28) and in Eq. (26).

$$u(t) \simeq U^T \Psi^\alpha(t)$$

$$\begin{aligned}
&\min \mathcal{J} \\
&\quad = \int_0^1 \mathcal{F}\left(t, C^T P^{(\nu,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)\right) \mathrm{d}t, \\
&C^T \Psi^\alpha(t) \\
&\quad = \mathcal{G}\left(t, C^T P^{(\nu,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)\right), \\
&\qquad 0 < \nu, \quad t \in [0,1], \\
&S_j(t, C^T P^{(\nu,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)) \le 0, \\
&\qquad j = 1, \dots, s.
\end{aligned} \tag{29}$$

Using slack variables we can transform the inequality constraint into an equality constraint

$$S_j(t, C^T P^{(v,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)) + Z_j^2(t) = 0. \tag{30}$$

Using the product OM of F-BWFs Eq. (17) and extending $Z_j(t)$ by F-BWFs, we have

$$S_j(t, C^T P^{(v,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)) + z_j^T \tilde{z}_j \Psi^\alpha(t) = 0. \tag{31}$$

From the above, by subtracting Ψa(t) from equation (29), we obtain

$$\begin{aligned} &\min\ \mathcal{J} = \int_0^1 \mathcal{F}\Big(t, C^T P^{(v,\alpha)} \Psi^\alpha(t) + d^T \Psi^\alpha(t), U^T \Psi^\alpha(t)\Big) \mathrm{d}t, \\ &C^T = \mathcal{G}\Big(t, C^T P^{(v,\alpha)} + d^T, U^T\Big), \qquad 0 < v, \quad t \in [0,1], \\ &S_j(t, C^T P^{(v,\alpha)} + d^T, U^T) + z_j^T \tilde{z}_j = 0, \qquad j = 1, \ldots, s. \end{aligned} \tag{32}$$

Currently, by applying the Lagrange multipliers approach, we have

$$\bar{J} = \mathcal{J} + \Big(C^T - \mathcal{G}\Big(t, C^T P^{(v,\alpha)} + d^T, U^T\Big)\Big)\lambda + \Big(S_j(t, C^T P^{(v,\alpha)} + d^T, U^T) + z_j^T \tilde{z}_j\Big)\delta_j, \tag{33}$$

The following system provides the maximum requirements

$$\frac{\partial \bar{J}}{\partial C} = 0, \frac{\partial \bar{J}}{\partial U} = 0, \frac{\partial \bar{J}}{\partial Z} = 0, \frac{\partial \bar{J}}{\partial \lambda} = 0, \quad \frac{\partial \bar{J}}{\partial \delta_j} = 0, \quad j = 1, \ldots, s. \tag{34}$$

Using programs like MATLAB, we can find C and U and get the approximate answer to Equation (26).

5. Example Test Problems:

Several numerical examples are given in this part to show the effectiveness and dependability of the suggested approach. For all of the numerical calculations, MATLAB 2018a was used.

Example: Examine the following FOCP with the inequality restriction that [45] describes**.**

$$\begin{aligned} &\min\ \mathcal{J} = \int_0^1 \Big(x^2(t) + u^2(t) + 2t^{\frac{3}{2}}x(t) - 2(1 - t^{\frac{3}{2}})\Big)\, u(t))dt, \\ &{}_0^C D_t^{\frac{3}{2}} x(t) = \tfrac{3\sqrt{\pi}}{4}(x(t) - u(t)), \\ &x(0) = \dot{x}(0) = 0, \\ &x(t) \le 0, \\ &0 \le u(t) \le 1. \end{aligned} \tag{35}$$

The minimizing solutions for the state and control variables in this problem are

$x(t) = -t^{3/2}$ and $u(t) = 1 - t^{3/2}$,

$${}_0^C D_t^{\frac{3}{2}} x(t) = C^T \Psi^\alpha(t), \tag{36}$$

$$x(t) = C^T P^{(\frac{3}{2},\alpha)} \Psi^\alpha(t), u(t) = U^T \Psi^\alpha(t), \quad (37)$$

When it is necessary to calculate the vectors of the unknown coefficients C and U. After replacing equation (35), for equation (37) we obtain

$$\min \mathcal{J} = \int_0^1 \Big(\left(C^T P^{(\frac{3}{2},\alpha)} \Psi^\alpha(t) \right)^2 + \left(U^T \Psi^\alpha(t) \right)^2 + 2t^{\frac{3}{2}} \left(C^T P^{(\frac{3}{2},\alpha)} \Psi^\alpha(t) \right) - 2\left(1 - t^{\frac{3}{2}}\right) \left(U^T \Psi^\alpha(t) \right) \Big) \mathrm{d}t,$$

Subject

$$C^T \Psi^\alpha(t) = \frac{3\sqrt{\pi}}{4} \left(C^T P^{(\frac{3}{2},\alpha)} \Psi^\alpha(t) - U^T \Psi^\alpha(t) \right),$$

Additionally, we have $0 \leq u(t) \leq 1$ and $x(t) \leq 0$. Thus, employing slack variables, we've

X(t) + s^2(t) = 0, uX(t) + s^2(t) = 0, u(t) - z^2(t) = 0; u(t) + w^2(t) = 1

Consequently,

$$C^T P^{(\frac{3}{2},\alpha)} \Psi^\alpha(t) + S^T \Psi^\alpha(t) \Psi^\alpha(t)^T S = 0,$$
$$U^T \Psi^\alpha(t) - Z^T \Psi^\alpha(t) \Psi^\alpha(t)^T Z = 0,$$
$$U^T \Psi^\alpha(t) + W^T \Psi^\alpha(t) \Psi^\alpha(t)^T W = E^T \Psi^\alpha,$$

Table 1 With k = 1, M = 1, and k = 1, the performance indicator value for the example in the case of M = 2

k = 1, M = 1		**k = 1, M = 2**	
a	**J**	**a**	**J**
1	- 0.696326654061	1	- 0.6999603188947
1.1	- 0.696776379794	1.1	- 0.6999545485260
1.2	- 0.6999588095082	1.2	- 0.698768230495
1.3	- 0.699485668624	1.3	- 0.6999752536707
1.4	- 0.699871201375	1.4	- 0.6999921239567
1.5	- 0.699999909999	1.5	-0.69999099999

Table 2 For example, the anticipated values of J for v = a = 3/2 using the F-BWFs approach and the Ritz method

Methods	J
Zeta penalty using the Ritz technique	-0.899989
m = 0.1, n = k = 5	
Present method	
Present method	-0.699969

Table 3 The state and control variable absolute errors with k = 1, M = 1, and m = a = 3/2 for example

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
ex	1.6 E-14	9.6 E-14	2.3 E-13	4.2 E-13	6.5 E-13	9.0 E-13	1.1 E-12	1.4 E-12	1.7 E-12
eu	2.5 E-12	2.4 E-12	2.3 E-12	2.0 E-12	1.8 E-12	1.5 E-12	1.2 E-12	9.2 E-13	5.5 E-13

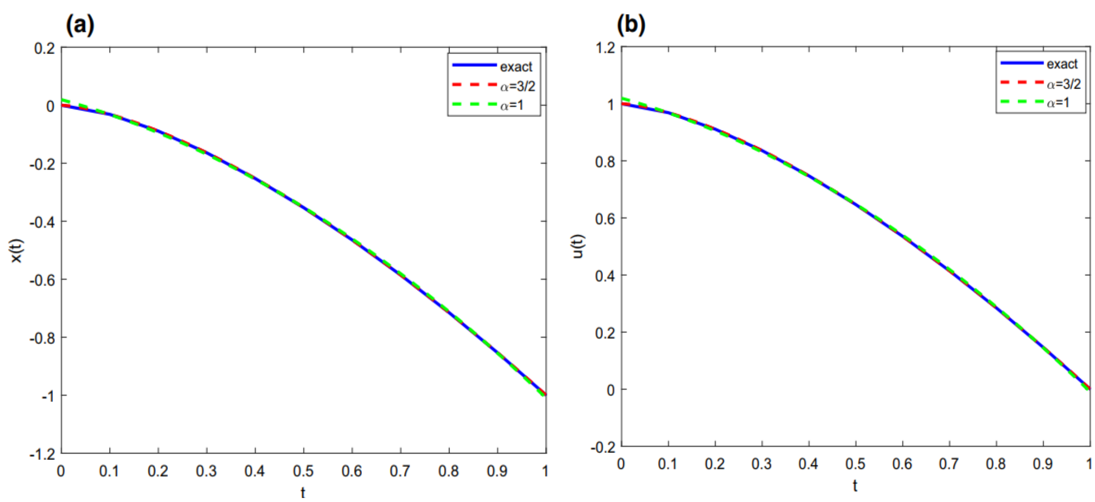

Fig. 1 Behavior of the control variables u(t) and x(t), when k = 1, M = 2, and a = 1, 3/2

6. Conclusion:

This paper proposes a new and efficient numerical technique to deal with FOCPs with inequality constraints based on F-BWFs and Lagrange multipliers. Using F-BWFs to expand the solution is the main idea. The basis of F-BWFs, the numerical results clearly show that the recommended strategy provides superior accuracy and efficiency #Innovation #NumericalMethods #Optimization

References:

[1] R. L. Bagley and P. J. Torvik, “Fractional calculus in the transient analysis of viscoelastically damped structures,” *AIAA journal*, vol. 23, no. 6, pp. 918–925, 1985.

[2] V. V Kulish and J. L. Lage, “Application of fractional calculus to fluid mechanics,” *J. Fluids Eng.*, vol. 124, no. 3, pp. 803–806, 2002.

[3] K. B. Oldham, “Fractional differential equations in electrochemistry,” *Advances in Engineering software*, vol. 41, no. 1, pp. 9–12, 2010.

[4] M. S. Dahaghin and H. Hassani, “An optimization method based on the generalized polynomials for nonlinear variable-order time fractional diffusion-wave equation,” *Nonlinear Dyn*, vol. 88, pp. 1587–1598, 2017.

[5] A. H. Bhrawy and M. A. Zaky, “Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations,” *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 1100–1117, 2017.

[6] K. Diethelm and G. Walz, “Numerical solution of fractional order differential equations by extrapolation,” *Numer Algorithms*, vol. 16, pp. 231–253, 1997.

[7] K. Diethelm, N. J. Ford, and A. D. Freed, “A predictor-corrector approach for the numerical solution of fractional differential equations,” *Nonlinear Dyn*, vol. 29, pp. 3–22, 2002.

[8] A. M. A. El-Sayed and M. Gaber, “The Adomian decomposition method for solving partial differential equations of fractal order in finite domains,” *Phys Lett A*, vol. 359, no. 3, pp. 175–182, 2006.

[9] L. Galeone and R. Garrappa, “On multistep methods for differential equations of fractional order,” *Mediterranean Journal of Mathematics*, vol. 3, pp. 565–580, 2006.

[10] Z. Odibat, S. Momani, and H. Xu, “A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations,” *Appl Math Model*, vol. 34, no. 3, pp. 593–600, 2010.

[11] M. Lakestani, M. Dehghan, and S. Irandoust-Pakchin, “The construction of operational matrix of fractional derivatives using B-spline functions,” *Commun Nonlinear Sci Numer Simul*, vol. 17, no. 3, pp. 1149–1162, 2012.

[12] R. Garrappa and M. Popolizio, “On accurate product integration rules for linear fractional differential equations,” *J Comput Appl Math*, vol. 235, no. 5, pp. 1085–1097, 2011.

[13] M. ur Rehman and R. A. Khan, “The Legendre wavelet method for solving fractional differential equations,” *Commun Nonlinear Sci Numer Simul*, vol. 16, no. 11, pp. 4163–4173, 2011.

[14] M. H. Heydari, M. R. Hooshmandasl, F. M. M. Ghaini, and F. Fereidouni, “Two-dimensional Legendre wavelets for solving fractional Poisson equation with Dirichlet boundary conditions,” *Eng Anal Bound Elem*, vol. 37, no. 11, pp. 1331–1338, 2013.

[15] R. B. Martin, “Optimal control drug scheduling of cancer chemotherapy,” *Automatica*, vol. 28, no. 6, pp. 1113–1123, 1992.

[16] G. Feichtinger, R. F. Hartl, and S. P. Sethi, “Dynamic optimal control models in advertising: recent developments,” *Manage Sci*, vol. 40, no. 2, pp. 195–226, 1994.

[17] P. Howlett, “The optimal control of a train,” *Ann Oper Res*, vol. 98, pp. 65–87, 2000.

[18] J. Vittek, P. Butko, B. Ftorek, P. Makyš, and L. Gorel, “Energy near-optimal control strategies for industrial and traction drives with ac motors,” *Math Probl Eng*, vol. 2017, 2017.

[19] B. K. Sarabi, M. Sharma, D. Kaur, and N. Kumar, “An optimal control based technique for generating desired vibrations in a structure,” *Iranian Journal of Science and Technology, Transactions of Electrical Engineering*, vol. 41, pp. 219–228, 2017.

[20] W. Li, S. Wang, and V. Rehbock, “Numerical solution of fractional optimal control,” *J Optim Theory Appl*, vol. 180, pp. 556–573, 2019.

[21] G. W. Swan, “Role of optimal control theory in cancer chemotherapy,” *Math Biosci*, vol. 101, no. 2, pp. 237–284, 1990.

[22] O. P. Agrawal, “A general formulation and solution scheme for fractional optimal control problems,” *Nonlinear Dyn*, vol. 38, pp. 323–337, 2004.

[23] S. A. Yousefi, A. Lotfi, and M. Dehghan, “The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems,” *Journal of Vibration and Control*, vol. 17, no. 13, pp. 2059–2065, 2011.

[24] M. Alipour, D. Rostamy, and D. Baleanu, “Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices,” *Journal of Vibration and Control*, vol. 19, no. 16, pp. 2523–2540, 2013.

[25] M. Alipour, D. Rostamy, and D. Baleanu, “Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices,” *Journal of Vibration and Control*, vol. 19, no. 16, pp. 2523–2540, 2013.

[26] A. Nemati, S. Yousefi, F. Soltanian, and J. S. Ardabili, “An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix,” *Asian J Control*, vol. 18, no. 6, pp. 2272–2282, 2016.

[27] J. Sabouri, S. Effati, and M. Pakdaman, “A neural network approach for solving a class of fractional optimal control problems,” *Neural Process Lett*, vol. 45, pp. 59–74, 2017.

[28] M. Behroozifar and N. Habibi, “A numerical approach for solving a class of fractional optimal control problems via operational matrix Bernoulli polynomials,” *Journal of Vibration and Control*, vol. 24, no. 12, pp. 2494–2511, 2018.

[29] M. Behroozifar and N. Habibi, “A numerical approach for solving a class of fractional optimal control problems via operational matrix Bernoulli polynomials,” *Journal of Vibration and Control*, vol. 24, no. 12, pp. 2494–2511, 2018.

[30] P. Rahimkhani and Y. Ordokhani, “Generalized fractional-order Bernoulli–Legendre functions: an effective tool for solving two-dimensional fractional optimal control problems,” *IMA J Math Control Inf*, vol. 36, no. 1, pp. 185–212, 2019.

[31] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, “A numerical solution for fractional optimal control problems via Bernoulli polynomials,” *Journal of Vibration and Control*, vol. 22, no. 18, pp. 3889–3903, 2016.

[32] P. Rahimkhani, Y. Ordokhani, and E. Babolian, “An efficient approximate method for solving delay fractional optimal control problems,” *Nonlinear Dyn*, vol. 86, pp. 1649–1661, 2016.

[33] H. Mirinejad and T. Inanc, “An RBF collocation method for solving optimal control problems,” *Rob Auton Syst*, vol. 87, pp. 219–225, 2017.

[34] K. Rabiei, Y. Ordokhani, and E. Babolian, “The Boubaker polynomials and their application to solve fractional optimal control problems,” *Nonlinear Dyn*, vol. 88, pp. 1013–1026, 2017.

[35] H. Hassani, J. A. T. Machado, and E. Naraghirad, “Generalized shifted Chebyshev polynomials for fractional optimal control problems,” *Commun Nonlinear Sci Numer Simul*, vol. 75, pp. 50–61, 2019.