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# Stability and Stabilization of Integro-Differential Perturbed Nonlinear System

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**A B S T R A C T**

The aim of this paper is to presented a problem formulation which is Integro-Differential perturbed of main matrix of nonlinear systems involving nonlinear functional one of them is an integro nonlinear function. The issue is to study the stability and stabilization of behavior solution, whetre in stabilization the system forced by feedback control function to grantee the exponentialy stable. Moreover the probability density function appered as a function in mailed solution of proposal problem formulation, some interesting examples explained the stability result which included the sufficient conditions of exponentialy stable and stabilization

.***Keywords***:

## 1. Introduction

The field of control and systems is currently one of the most important topics that play a good role in simplifying some systems. The control is involved in non-linear systems and the interpretation of complex phenomena, which is great benefit in modernizing human civilization day after day [1].

Then, Li and *et al.* are studying of the global stability problem for feedback control systems of impulsive fractional differential equations on networks [2]. Another direction studied by Qasim and *et* for some classes with composition FOCPs as in [3]. The Stability and destabilization of fractional oscillators via a delayed feedback control have been considered by Čermák *et al.* in [4]. The Mittag-Leffler stabilization of fractional-order

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nonlinear systems with unknown control coefficients is verified and examined by Wang in [5]. Hasan S. Q. studied stability and uniform stability of single and multi-control integro-differential inequality in [6], [7].

The main objective of this work is to study integro-differential nonlinear systems with an ordinary derivative which described as a control system. The systems that are unstable were examined, then a feedback process from the control function is presented for stability . We are investigated and demonstrated the local stability with complete accuracy for nonlinear systems.

The outline of the paper is organized as follows: Section 2, present some basic preliminaries concept and some auxiliary definitions of problem formulation was presented as new time, and provided the results that have discovered which focused on the Stability problem of integro-differential nonlinear feedback control system. The control function is used to be bounded that make the system is controller for activity of uncertainty function that observed in component of system.

## 2. Problem Formulation :

Consider the following integro-differential nonlinear equation

$$\dot{x}(t) = (A + \Delta A)x(t) + f_1(t, x(t)) \int_0^t f_2\big(\tau, x(\tau)\big) d\tau \qquad (1)$$

$$x(0) = x_0$$

**Lemma (2.1)**: The integro-differential nonlinear problem (1) is equivalent to the double integral equation

$$x(t) = x_0 + (A + \Delta A) \int_0^t x(\tau) d\tau + \int_0^t f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big) ds\, d\tau \qquad (2)$$

**Proof:**

Let us assume that $x \in C$ satisfies the integro-differential nonlinear problem (1). So to prove equation (2) as follows:

Appling the integral operator${}_0I_t$ to both sides of (1), we have

$${}_0I_t\dot{x}(t) = (A+\Delta A)\,{}_0I_t x(t) + {}_0I_t f_1\big(\tau, x(\tau)\big)\int_0^{\tau} f_2\big(s, x(s)\big)ds \qquad (3)$$

we have that,

$$x(t) - x_0 = (A+\Delta A)\int_0^t x(\tau)d\tau + \int_0^t f_1\big(\tau, x(\tau)\big)\int_0^{\tau} f_2\big(s, x(s)\big)ds\, d\tau$$

$$x(t) = x_0 + (A+\Delta A)\int_0^t x(\tau)d\tau + \int_0^t f_1\big(\tau, x(\tau)\big)\int_0^{\tau} f_2\big(s, x(s)\big)ds\, d\tau \qquad (4)$$

Appling the derivative operator ${}_0D_t$ of (2) we have that

$${}_0D_t x(t) = {}_0D_t x_0 + (A+\Delta A)\,{}_0D_t\,{}_0I_t x(t) + {}_0D_t\,{}_0I_t f_1\big(\tau, x(\tau)\big)\int_0^{\tau} f_2\big(s, x(s)\big)ds$$

(5)

We obtain,

$$\dot{x}(t) = (A+\Delta A)x(t) + f_1(t, x(t))\int_0^t f_2\big(\tau, x(\tau)\big)d\tau \qquad (6)$$

**<u>Remark(2.2):</u>** The Laplace transform of the one-sided stable probability density

$$\rho(\eta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\eta^{-n-1}\frac{\Gamma(n+1)}{n!}\sin(n\pi), \quad \eta \in (0,\infty)$$

is given by

$$\int_0^{\infty} e^{-st}\rho(\eta)d\,\eta = e^{-s}\text{, where } s > 0. \qquad (7)$$

furthermore, for any $0 \le \delta \le 1$, we have

$$\int_0^{\infty}\frac{1}{\eta^{\delta}}\rho(\eta)d\eta = 1\,\eta$$

(8)

**<u>Theorem (2.3)</u>:**

If (2) is satisfied, then

$x(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds)d\tau$ for $t \in [0, b]$,

where

$$\widehat{U}(t) = \int_0^\infty \hat{\xi}(\eta)\widehat{T}(t\eta)xd\eta \quad \forall x \in \mathrm{X} \tag{9}$$

and

$$\widehat{W}(t) = \int_0^\infty \eta\hat{\xi}(\eta)\widehat{T}(t\eta)xd\eta \quad \forall x \in \mathrm{X} \tag{10}$$

for $\hat{\xi}(\eta) = (-\eta^{-2}\rho(\eta^{-1}))$ (11)

is the probability density function defined on $(0, \infty)$, that is, $\xi(\eta) \geq 0$ for $\eta \in (0, \infty)$ and $\int_0^\infty \xi(\eta)xd\eta = 1$

**proof:**

Let $s > 0$. Take Laplace transform of both sides of (2), we get,

$$\mathcal{L}\{x(t)\}(s) = x_0\mathcal{L}\{1\} + (A + \Delta A)\mathcal{L}\left\{\int_0^t x(\tau)d\tau\right\} +$$

$$\mathcal{L}\left\{\int_0^t f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\, d\tau\right\}$$

Hence

$$X(s) = \frac{1}{s}x_0 + (A + \Delta A)X(s) + \mathcal{L}\left\{\int_0^t f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\, d\tau\right\}$$

$$X(s) = \frac{1}{s}x_0 + (A + \Delta A)X(s) + \mathcal{L}\left\{\int_0^t f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\, d\tau\right\} \tag{12}$$

Therefore,

$X(s) =$

$(sI - (A + \Delta A))^{-1}x_0 +$

$$\left(sI-(A+\Delta A)\right)^{-1}\mathcal{L}\left\{\int_0^t f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\,d\tau\right\}$$

$$X(s)=\mathcal{L}\{\hat{T}(t)\}x_0+\mathcal{L}\{\hat{T}(t)\}\,\mathcal{L}\left\{\int_0^t f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\,d\tau\right\}$$

$$X(s)=$$

$$\int_0^\infty e^{-st}\,\hat{T}(t)x_0dt+\int_0^\infty e^{-st}\,\hat{T}(t)dt\mathcal{L}\left\{\int_0^t f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\,d\tau\right\} \tag{13}$$

$$X(s)=\int_0^\infty e^{-st}\,\hat{T}(t)x_0dt+$$

$$\int_0^\infty e^{-st}\,\hat{T}(t)\big(\int_0^\infty e^{-s\tau} f_1\big(\tau,x(\tau)\big)\big(\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau\big)dt$$

$$X(s)=\int_0^\infty -\frac{1}{s}\hat{T}(t)x_0\frac{d}{dt}e^{-st}dt+$$

$$\int_0^\infty\int_0^\infty e^{-st}\,\hat{T}(t)e^{s\tau}f_1\big(\tau,x(\tau)\big)\big(\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau dt \tag{14}$$

By (7), equation (14) becomes:

$$X(s)=\int_0^\infty -\frac{1}{s}\hat{T}(t)x_0\left(\int_0^\infty\frac{d}{dt}e^{-st\sigma}\rho(\sigma)d\sigma\right)dt+$$

$$\int_0^\infty\int_0^\infty\left(\int_0^\infty\frac{d}{dt}e^{-st\sigma}\rho(\sigma)d\sigma\right)\hat{T}(t)e^{-s\tau}f_1\big(\tau,x(\tau)\big)\big(\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau dt \tag{15}$$

$$X(s)=\int_0^\infty\int_0^\infty\sigma\,\hat{T}(t)x_0e^{-st\sigma}\rho(\sigma)d\sigma dt+$$

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-st\sigma}\rho(\sigma)\,\hat{T}(t)e^{-s\tau}f_1\big(\tau,x(\tau)\big)\big(\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\sigma d\tau dt \tag{16}$$

By $t\sigma=\theta\ \Rightarrow dt=\frac{1}{\sigma}d\theta$, we get that

$$X(s)=\int_0^\infty\int_0^\infty\hat{T}\left(\frac{\theta}{\sigma}\right)e^{-s\theta}x_0\rho(\sigma)d\sigma d\theta+$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{1}{\sigma}e^{-s(\theta+\tau)}\rho(\sigma)\,\hat{T}\left(\frac{\theta}{\sigma}\right)f_1\big(\tau,x(\tau)\big)\big(\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\sigma d\tau d\theta \tag{17}$$

Now take $\theta + \tau = t$, we have that,

$$\begin{cases}\theta \to 0 \Rightarrow \ t \to \tau \\ \theta \to \infty \Rightarrow t \to \infty\end{cases}, \begin{cases}0 < \tau < \infty \\ \tau < t < \infty\end{cases} \Rightarrow \ 0 < \tau < t$$

and

$$\begin{cases}0 < \tau < \infty \\ \tau < t < \infty\end{cases} \Rightarrow \ 0 < t < \infty$$

Then, (17), becomes:

$$X(s) = \int_0^\infty e^{-s\theta} \left(\int_0^\infty \hat{T}\left(\frac{\theta}{\sigma}\right) \rho(\sigma) x_0 d\sigma\right) d\theta +$$

$$\int_{t=0}^\infty e^{-st} \ \left(\int_{\sigma=0}^\infty \int_{\tau=0}^t \frac{1}{\sigma} \rho(\sigma) \hat{T}\left(\frac{t-\tau}{\sigma}\right) f_1\big(\tau, x(\tau)\big)\left(\int_0^\tau f_2\big(s, x(s)\big) ds\right) d\tau d\sigma\right) dt$$

$$X(s) = \int_{t=0}^\infty e^{-st} \left\{\int_0^\infty \hat{T}\left(\frac{t}{\sigma}\right) \rho(\sigma) x_0 d\sigma + \ \left(\int_{\sigma=0}^\infty \int_{\tau=0}^t \frac{1}{\sigma} \rho(\sigma)\right.\right.$$

$$\left.\left.\hat{T}\left(\frac{t-\tau}{\sigma}\right) f_1\big(\tau, x(\tau)\big)\left(\int_0^\tau f_2\big(s, x(s)\big) ds\right) d\tau d\sigma\right\} dt \tag{18}$$

By using inverse Laplace transform on (18), we obtain

$$x(t) = \int_0^\infty \hat{T}\left(\frac{t}{\sigma}\right) \rho(\sigma) x_0 d\sigma$$

$$+ \int_{\sigma=0}^\infty \int_{\tau=0}^t \frac{1}{\sigma} \rho(\sigma) \hat{T}\left(\frac{t-\tau}{\sigma}\right) f_1\big(\tau, x(\tau)\big)\left(\int_0^\tau f_2\big(s, x(s)\big) ds\right) d\tau d\sigma \tag{19}$$

Now if we take,

$$\frac{1}{\sigma} = \eta \ \Rightarrow \ \sigma = \eta^{-1} \ and \ d\sigma = -\eta^{-2} d\eta.$$

Equation (19) becomes

$$x(t) = \int_0^\infty \hat{T}(t\eta)\big(-\eta^{-2} \rho(\eta^{-1})\big) x_0 d\eta$$

$$+ \int_0^\infty \int_0^t \eta \hat{T}\big((t-\tau)\eta\big)\big(-\eta^{-2} \rho(\eta^{-1})\big) f_1\big(\tau, x(\tau)\big)\left(\int_0^\tau f_2\big(s, x(s)\big) ds\right) d\tau d\eta$$

$x(t) = \int_0^\infty \hat{T}(t\eta)\, \xi(\eta)\, x_0$

$+ \int_0^\infty \int_0^t \eta \hat{T}\big((t-\tau)\eta\big)\, \xi(\eta) f_1\big(\tau, x(\tau)\big), \big(\int_0^\tau f_2\big(s, x(s)\big)ds\big)d\tau d\eta$

where, $\hat{\xi}(\eta) = \big(-\eta^{-2}\rho(\eta^{-1})\big),\ \eta \in (0, \infty)$.

therefore, $x(t) = \hat{U}(t)x_0 + \int_0^t \hat{W}(t-\tau)(f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big)ds)d\tau$

for $t \in [0, b]$,

**Definition (2.4):**

A function $x \in C([0, b], X)$ is said to be a mild solution to the initial value problem (1) if $x$ satisfies

$x(t) = \hat{U}(t)x_0 + \int_0^t \hat{W}(t-\tau)(f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big)ds)d\tau$ for $t \in [0, b]$,

Were the operators $\{\hat{U}(t)\}_{t\in[0,b]}$ and $\{\hat{W}(t)\}_{t\in[0,b]}$ are defined in (9) and (10) respectively.

**Lemma (2.5)**

$\hat{U}(t)$ and $\hat{W}(t)$ are linear bounded operator, for any fixed $t \in [0, b]$,

**Proof:**

For any fixed $t \in [0, b]$, since $\hat{T}(t)$ and $\hat{U}(t)$ and $\hat{W}(t)$ are linear operators. also for any $0 \leq \delta \leq 1$ we have that

$$\int_0^\infty \eta^\delta \hat{\zeta}(\eta) d\eta = \int_0^\infty -\eta^\delta \eta^{-2} \rho(\eta^{-1})\, d\eta = \int_0^\infty -\eta^{\delta-2} \rho(\eta^{-1})\, d\eta \qquad (20)$$

Now from variable below, we have that

$\sigma = \eta^{-1} \Rightarrow \eta = \sigma^{-1} \Rightarrow d\eta = -\sigma^{-2} d\sigma.$

So, equation (20) implies that

$$\int_0^\infty \eta^\delta \hat{\zeta}(\eta)d\eta = \int_0^\infty \sigma^{-\delta}\rho(\sigma)\, d\sigma = \int_0^\infty \frac{1}{\sigma^\delta}\rho(\sigma)\, d\sigma\,. \quad (21)$$

But $0 \leq \delta \leq 1$, from (10), we have that

$$\int_0^\infty \eta^\delta \hat{\zeta}(\eta)d\eta = \ = 1 \quad (22)$$

In particular, for $\delta = 1$, we get that

$$\int_0^\infty \eta^\delta \hat{\zeta}(\eta)d\eta = 1$$

For any $x \in X$, we have that

$$\left\|\hat{U}(t)x\right\| = \left\|\int_0^\infty \hat{\xi}(\eta)\,\hat{T}(t\eta)xd\eta\right\| \leq \underbrace{\sup_{t\in[0,\infty)}\left\|\hat{T}(t)\right\|\|x\|}_{=M}\underbrace{\int_0^\infty \hat{\zeta}(\eta)d\eta}_{1}$$

$= M\|x\|$, hence $\frac{\|\hat{U}(t)x\|}{\|x\|} \leq M$,

Implies that, $\left\|\hat{U}(t)x\right\|_{\beta(x)} \leq M$ (23)

also,

$$\left\|\hat{W}(t)x\right\| = \left\|\int_0^\infty \eta\hat{\xi}(\eta)\hat{T}(t\eta)xd\eta\quad\right\| \leq \underbrace{\sup_{\tau\in[0,\infty)}\left\|\hat{T}(t)\right\|\|x\|}_{=M}\underbrace{\int_0^\infty \eta\hat{\zeta}(\eta)d\eta}_{1}$$

$= M\|x\|$, implies that $\frac{\|W(t)x\|}{\|x\|} \leq M$,

hence $\left\|\hat{W}(t)x\right\|_{\beta(x)} \leq M$ (24)

From (23) and (24), we obtain $\{U(t)\}$ and $\{\hat{W}(t)$are linear bounded.

**Theorem (2.6):**

Consider the following integro-differential perturbed nonlinear equation (1) satisfied the following conditions:

(1) $f_1\big(t,x(t)\big)$ and $f_2\big(t,x(t)\big)$ are of Caratheodory; that is, for any $x \in X, f_1\big(t,x(t)\big)$ and $f_2\big(t,x(t)\big)$ are strongly measurable with respect to $t \in [0,b]$ also $f_1\big(t,x(t)\big)$ and $f_2\big(t,x(t)\big)$ are continuous with respect to x ∈ X.

(2) $\left\| f_1\big(t,x(t)\big)\int_0^\tau f_2\big(s,x(s)\big)ds - f_1\big(t,y(t)\big)\int_0^\tau f_2\big(s,y(s)\big)ds \right\| \le L\|x - y\| \forall\, x,y \in X, \forall\, t \in [0,b]$ and $L = L_1 b + L_2 b M_2$.

(3) $\left\|f_1\big(t,x(t)\big)\right\| \le M_1$ and $\left\|f_2\big(t,x(t)\big)\right\| \le M_2$, where $M_1, M_2 > 0$.

Then (1), has a unique mild solution provided that the constant $\Omega = MLb$

Such that $0 \le \Omega \le 1.$ (25)

**<u>Proof</u>**:

Let $Kx(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds)d\tau$

for $t \in [0,b]$, we have that

$$\|Kx(t) - Ky(t)\| = \left\| \int_0^t \widehat{W}(t-\tau)(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds)d\tau - \int_0^t \widehat{W}(t-\tau)(f_1\big(\tau,y(\tau)\big)\int_0^\tau f_2\big(s,y(s)\big)ds)d\tau \right\| \le$$

$$\int_0^t \left\|\widehat{W}(t)x\right\|_{\beta(x)} \left\| f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds - \big(f_1\big(\tau,y(\tau)\big)\int_0^\tau f_2\big(s,y(s)\big)ds\big)\right\| d\tau \le$$

$$\int_0^t \left\|\widehat{W}(t)x\right\|_{\beta(\mathbb{X})} \left\| f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds - f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,y(s)\big)ds + f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,y(s)\big)ds - f_1\big(\tau,y(\tau)\big)\int_0^\tau f_2\big(s,y(s)\big)ds \right\| d\tau$$

$$\le \int_0^t \left\|\widehat{W}(t)x\right\|_{\beta(\mathbb{X})} (M_1 L_1 b + L_2 b M_2)\|x(\tau) - y(\tau)\| d\tau$$

which implies from Lemma (2.5) that

$\|Kx(t) - Ky(t)\| \leq ML\,\text{Sup}_{\tau\in[0,b]}\|x(\tau) - y(\tau)\| \int_0^t d\tau$

$= ML\,\text{Sup}_{\tau\in[0,b]}\|x(\tau) - y(\tau)\|[\tau]_{\tau=0}^t$

$= ML\|x - y\|t \leq MLb\|x - y\|\;\Omega\|x - y\|$

implies that $\text{Sup}_{t\in[0,b]}\|Kx(t) - Ky(t)\| \leq \Omega\|x - y\|$ (26)

**2.1 <u>Stability of the integro-differential perturbed nonlinear system</u>**

**<u>Definition(2.1.7):</u>** If the System (1) hold the following condition,

$\|x(t)\| \leq Ke^{-\omega t}\|x_0\| \;\forall\, t \in [0,b]$. for every initial state $x_0 \in \mathbb{X}$, with $K > 0$ and $\omega > 0$, is called exponentially stable.

**<u>Lemma(2.1.8),[2]:</u>** If the inequality satisfied for $t > t_0,\; 0 < u(t) < k + \int_{t_0}^t (l + \beta u(t))\,dt$, for u(t) a continuous function, where $k, l$ and $\beta$ are constants such that $k, l \geq 0$ and $\beta > 0$. Then $u(t) < \frac{1}{\beta}\left(e^{\beta(t-t_0)}\right) + ke^{\beta(t-t_0)} \quad \forall\; t > t_0$ .

**<u>Remark (2.1.9):</u>**

Consider the following perturbed linear system

$\dot{x}(t) = (A + \Delta A)x(t)$ (27)

$x(0) = x_0$ (28)

Has a solution given by:

$\widehat{U}(t)x = \int_0^\infty \hat{\xi}(\eta)\,\hat{T}(t\eta)x d\eta \quad \forall x \in \mathbb{X}.$ (29)

Now, let

$$\left\|\hat{T}(t\eta)\right\| \leq \widetilde{\mathcal{M}} e^{-\tilde{\mu}t} \qquad \forall\ t \in [0,b],\ \forall \eta \in (0,\infty) \tag{30}$$

where $\widetilde{\mathcal{M}}$, $\tilde{\mu}$ >0.

From (29) and (30), we have that

$$\left\|\widehat{U}(t)x\right\|_{\beta(x)} \leq \int_0^\infty \xi\hat{\xi}(\eta)\,\hat{T}(t\eta)d\eta \leq \int_0^\infty \hat{\xi}(\eta)\,\widetilde{\mathcal{M}} e^{-\tilde{\mu}t} d\eta$$

$$\leq \widetilde{\mathcal{M}} e^{-\tilde{\mu}t} \underbrace{\int_0^\infty \xi(\eta)\, d\eta}_{1} = \widetilde{\mathcal{M}} e^{-\tilde{\mu}t}$$

So, $\|U(t)x\|_{\beta(x)} \leq \widetilde{\mathcal{M}} e^{-\tilde{\mu}t} \quad \forall\ t \in [0,b]$ (31)

**Theorem (2.1.10):**

If (30) and condition (2) of theorem(2.5) are satisfied, and the constants $\widetilde{\mathcal{M}}, \tilde{\mu}$ and $L$ satisfy the following

$$\tilde{\mu} - L\widetilde{\mathcal{M}} > 0, \tag{32}$$

then the solution of the integro-differential perturbed nonlinear system (1) is exponentially stable.

**Proof:**

We have that x ∈ C([0, b],X) satisfies (1), then x can be written as follow:

$x(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds)d\tau$ for $t \in [0,b]$.

From (30) and (31), we have:

$$\left\|\widehat{W}(t)x\right\|_{\beta(x)} \leq \int_0^\infty \eta\hat{\xi}(\eta)\hat{T}(t\eta)d\eta$$

$$\leq \int_0^\infty \eta\hat{\xi}(\eta)\widetilde{\mathcal{M}} e^{-\tilde{\mu}t} d\eta \quad = \widetilde{\mathcal{M}} e^{-\tilde{\mu}t} \underbrace{\int_0^\infty \eta\hat{\xi}(\eta)d\eta}_{1}$$

So, $\left\|\widehat{W}(t)x\right\|_{\beta(\mathbb{X})} \leq \widetilde{\mathcal{M}}e^{-\widetilde{\mu}t}\ \forall\ t \in [0,b]$.

(33)

Substitute $y = 0$ in Lipschitz condition gives for any $x \in X$ and any $t \in [0,b]$, the following:

$$\left\|f_1\big(t,x(t)\big)\int_0^t f_2\big(s,x(s)\big)ds\right\| = L\|x\| \tag{34}$$

Then, by (31), (33) and (34), we obtain that,

$$\|x(t)\| \leq \widetilde{\mathcal{M}}e^{-\widetilde{\mu}t}\|x_0\| + L\widetilde{\mathcal{M}}\int_0^t \|x(\tau)\|\, e^{-\widetilde{\mu}(t-\tau)}d\tau$$

$$\leq \widetilde{\mathcal{M}}e^{-\widetilde{\mu}t}\|x_0\| + L\widetilde{\mathcal{M}}\sup_{\tau\in[0,b]}\int_0^t e^{-\widetilde{\mu}(t-\tau)}\,\|x(\tau)\|d\tau,$$

Since

$$0 \leq \tau \leq t \ \Rightarrow\ -t \leq -\tau \leq 0 \ \Rightarrow\ 0 \leq t-\tau \leq t \leq b$$

We have that,

$$\|x(t)\| \leq \widetilde{\mathcal{M}}e^{-\widetilde{\mu}t}\|x_0\| + L\widetilde{\mathcal{M}}\int_0^t e^{-\widetilde{\mu}(t-\tau)}\,\|x(\tau)\|d\tau. \tag{35}$$

Now, let function $\psi$ defined on $[0,b]$ as follows:

$$\psi(t) = e^{-\widetilde{\mu}t}\|x(t)\| \tag{36}$$

We obtain from (35), that

$$\psi(t) \leq \widetilde{\mathcal{M}}\|x_0\| + L\widetilde{\mathcal{M}}\int_0^t \psi(t)\,d\tau. \tag{37}$$

By Lemma(2.1.8), we obtain from (37) that

$$\psi(t) \leq \widetilde{\mathcal{M}}e^{(L\widetilde{\mathcal{M}})t}\|x_0\| \quad \forall\ t \in [0,b]$$

Therefore, by (35) implies that $\|x(t)\| \leq \widetilde{\mathcal{M}}e^{(L\widetilde{\mathcal{M}}-\widetilde{\mu})t}\|x_0\| \quad \forall\ t \in [0,b]$

Therefore, according to (32), the solution is exponentially stable.

## 2.2 Stabilization of the integro-differential perturbed nonlinear system

Consider the integro-differential perturbed nonlinear control system

$$\begin{cases} \dot{x} = (A + \Delta A)x(t) + f_1\big(t, x(t)\big) \int_0^t f_2\big(\tau, x(\tau)\big)d\tau + Bu(t), \quad t \in [0, b] \\ x(0) = x_0, \end{cases} \tag{38}$$

where $U$ be a separable reflexive Banach space(control space). the mild solution in definition(2.4) is well defined for every integrable control $u(t)$, we obtain

$$x(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)\Big(f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau +$$

$$\int_0^t \widehat{W}(t-\tau)Bu(\tau)d\tau \quad \text{for} \quad t \in [0, b]. \tag{39}$$

Let $u = V x$, where $V : \mathbb{X} \longrightarrow U$ is a bounded linear operator. Then, (39) becomes:

$$x(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)\Big(f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau +$$

$$\int_0^t \widehat{W}(t-\tau)BV\, x(\tau)d\tau \quad \text{for} \quad t \in [0, b]. \tag{40}$$

**Definition(2.2.11):** The integro-differential perturbed nonlinear system (1) is said to be stabilizable if $V : \mathbb{X} \longrightarrow U$ is a bounded linear operator satisfying $u(t) = V\, x(t)$ for any $t \in [0, b]$ such that integro-differential perturbed nonlinear control system (38) is stable.

**Lemma(2.2.12),[2]:** If the inequality $v(t) \le \beta + \int_0^t k(\tau)v(\tau)d\tau \quad \forall\, t \in [0, b]$, is satisfied with $\beta \ge 0$ and $b > 0$ and a nonnegatives continuous functions $v, k : [0, b] \longrightarrow [0, \infty)$ are bounded.

**Theorem (2.2.13):**

The integro-differential perturbed nonlinear control system (37) satisfied the following

$$\tilde{\mu} - \left(L + \|BV\|_{\beta(\mathbb{X})}\right)\widetilde{\mathcal{M}} > 0 \tag{41}$$

$\widetilde{\mathcal{M}}, \tilde{\mu}$ ,$L$ are constants and the bounded linear operators $B, V$, for $\eta \in (0, \infty)$, then the exponentially stabilizable hold for problem (38) .

**Proof:**

Consider the following operator, for $x_0 \in \mathbb{X}$

$$Y(t)x_0 = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)BV\, Y(\tau)x_0 d\tau \quad \text{for} \quad t \in [0, b]. \tag{42}$$

We have that,

$$\|Y(t)\|_{\beta(\mathbb{X})} \leq \widetilde{\mathcal{M}} e^{-\tilde{\mu}t} + \|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}} \int_0^t e^{-\tilde{\mu}(t-\tau)} \|Y(\tau)\| d\tau$$

and then, by proving of remark (2.9), we obtain

$$\|Y(t)\|_{\beta(\mathbb{X})} \leq \widetilde{\mathcal{M}} e^{\left(\|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}} - \tilde{\mu}\right)t} \quad \forall\, t \in [0, b]. \tag{43}$$

Then,

$$x(t) = \widehat{U}(t)x_0 + \int_0^t \widehat{W}(t-\tau)\left(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\right)d\tau +$$

$$\int_0^t \widehat{W}(t-\tau)BV\, x(\tau)d\tau + Y(t)x_0 - Y(t)x_0 +$$
$$\int_0^t Y(t-\tau)\left(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\right)d\tau -$$
$$+ \int_0^t Y(t-\tau)\left(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\right)d\tau$$

$$= \int_0^t Y(t-\tau)\left(f_1(\tau, x(\tau)) \int_0^\tau f_2(s, x(s))ds\right)d\tau + Y(t)x_0$$

$$+\int_0^t \widehat{W}(t-\tau)BV\,[x(\tau)-Y(\tau)x_0]d\tau$$

$$+\int_0^t[\widehat{W}(t-\tau)-Y(t-\tau)]\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau +$$

$$=\int_0^t Y(t-\tau)\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau + Y(t)x_0$$

$$+\int_0^t \widehat{W}(t-\tau)BV\,[x(\tau)-Y(\tau)x_0]d\tau$$

$$+\int_0^t[\widehat{W}(t-\tau)-Y(t-\tau)]\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau +$$

$$-\int_0^t\left[\int_0^t \widehat{W}(t-(\tau+r))BV\,Y(r)dr\right]\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau$$

Take variable $\theta = \tau + r$, we get

$$x(t)=\int_0^t Y(t-\tau)\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau + Y(t)x_0$$

$$+\int_0^t W(t-\tau)BV\,[x(\tau)-Y(\tau)x_0]d\tau$$

$$+\int_0^t[\int_0^\infty\ (\eta-1)\xi(\eta)\,\hat{T}((t-\tau)\eta)d\eta\ \ ]\,\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau$$

$$-\int_0^t\left[\int_\tau^t\ (t-\theta)W(t-\theta)\,BV\,Y(\theta-\tau)\eta)d\theta\right]\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau$$

$$\leq\int_0^t Y(t-\tau)\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau + Y(t)x_0$$

$$+\int_0^t W(t-\tau)BV\,[x(\tau)-Y(\tau)x_0]d\tau$$

$$-\int_0^t\left[\int_\tau^t\ (t-\theta)W(t-\theta)\,BV\,Y(\theta-\tau)\eta d\theta\right]\big(f_1\big(\tau,x(\tau)\big)\int_0^\tau f_2\big(s,x(s)\big)ds\big)d\tau$$

But we know that

$$\begin{cases}\tau\leq\theta\leq t\\ 0\leq\tau\leq t\end{cases}\ \Rightarrow\ 0\leq\tau\leq\theta \text{ and } \begin{cases}\tau\leq\theta\leq t\\ 0\leq\tau\leq t\end{cases}\ \Rightarrow\ 0\leq\theta\leq t.$$

So,

$$x(t) \leq \int_0^t Y(t-\tau)\Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau + Y(t)x_0$$

$$+\int_0^t W(t-\tau)BV\,[x(\tau) - Y(\tau)x_0]d\tau$$

$$-\int_0^t W(t-\theta)\,BV\left[\int_0^\theta Y(\theta-\tau)\ \Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau\right]d\theta$$

Implies that,

$$x(t) \leq \int_0^t Y(t-\tau)\Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau + Y(t)x_0$$

$$+\int_0^t W(t-\tau)BV\,\Big[x(\theta) - Y(\theta)x_0 -$$

$$\int_0^\theta Y(\theta-\tau)\ \Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau\Big]\,d\theta \qquad (44)$$

Let us define the function $\phi$ on $[0, b]$ as follow:

$$\phi(t) = x(t) - Y(t)x_0 - \int_0^t Y(t-\tau)\Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau. \qquad (45)$$

Then, (43) is equivalent to,

$$\phi(t) \leq \int_0^t W(t-\tau)BV\,\phi(\theta)d\theta$$

By (24) implies that

$$\|\phi(t)\| \leq \int_0^t M\|BV\|_{\beta(\mathbb{X})}\,\|\phi(\theta)\|d\theta. \qquad (46)$$

By Lemma (2.2.12), we obtain that

$\phi(t) = 0 \quad \forall\, t \in [0, b]$,that is,

$$x(t) = Y(t)x_0 + \int_0^t Y(t-\tau)\Big(f_1\big(\tau, x(\tau)\big)\int_0^\tau f_2\big(s, x(s)\big)ds\Big)d\tau. \qquad (47)$$

From (34) and (43), we have that

$$\|x(t)\| \leq \widetilde{\mathcal{M}}e^{\left(\|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}}-\widetilde{\mu}\right)t}\|x_0\| + L\widetilde{\mathcal{M}}\int_0^t e^{\left(\|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}}-\widetilde{\mu}\right)t}\|x(\tau)\|d\tau.$$

$$\psi(t) = e^{(\tilde{\mu}-\|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}})t}\|x(\tau)\|. \quad (48)$$

From (47), we obtain:

$$\psi(t) \le \widetilde{\mathcal{M}}\|x_0\| + L\widetilde{\mathcal{M}} \int_0^t \psi(\tau)d\tau. \quad (49)$$

since, the function $t \mapsto \psi(t)$ is continuous on $[0, b]$. From Lemma (2.1.8) and (48 ) that is

$$\psi(t) \le \widetilde{\mathcal{M}}\|x_0\|e^{L\widetilde{\mathcal{M}}t} \quad \forall\, t \in [0, b]$$

by (48) implies that,

$$\|x(t)\| \le \widetilde{\mathcal{M}}e^{(L\widetilde{\mathcal{M}}+\|BV\|_{\beta(\mathbb{X})}\widetilde{\mathcal{M}}-\tilde{\mu})t}\|x_0\| \quad \forall\, t \in [0, b];$$

Hence,

$$\|x(t)\| \le \widetilde{\mathcal{M}}e^{([L+\|BV\|_{\beta(\mathbb{X})}]\widetilde{\mathcal{M}}-\tilde{\mu})t}\|x_0\| \quad \forall\, t \in [0, b]; \quad (50)$$

By (41) we have that problem (38) is exponentially stabilizable.

**Example** (2.2):

We consider the following integro-differential perturbed nonlinear system:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + f_1(t, x(t)) \int_0^t f_2\big(\tau, x(\tau)\big)d\tau \;\; x \in l^\infty(\mathbb{N}) \\ \qquad\qquad x(0) = x_{0,} \end{cases} \quad (51)$$

where $X = l^\infty(\mathbb{N})$ is an infinite matrix defined from $l^\infty(\mathbb{N})$ to itself by

$(Ax)_i = \sum_{j=1}^{\infty} a_{ij}x_j$ such that for any $x \in l^\infty(\mathbb{N})$,we have that

$$\sum_{j=1}^{\infty}\left|a_{ij}x_j\right| < \infty \quad \forall i \in \mathbb{N} \quad (52)$$

and $sup\left\{\sum_{j=1}^{\infty}\left|a_{ij}\right| \;\right\} < \infty,$ (53)

also $(\Delta Ax)_i = \sum_{j=1}^{\infty} \epsilon_{ij} x_j$ ,$0 < \epsilon_{ij} < 1, \forall i \in \mathbb{N}$ (54)

$f_1, f_2$ are nonlinear functions defined on $[0,1] \times l^{\infty}(\mathbb{N})$ by

$$f_1\big(t, x(t)\big) \int_0^t f_2\big(\tau, x(\tau)\big) d\tau \; = \frac{\cos(\|x\|_\infty)}{3e^t} \int_0^t \frac{\cos(\|x\|_\infty)}{3e^\tau} d\tau$$

From (52),(53) and (54) , $A + \Delta A$ is a bounded linear operator on $l^{\infty}(\mathbb{N})$ and its norm is given by:

$$\|A + \Delta A\, \|_\infty = sup\{\sum_{j=1}^{\infty} |a_{ij}| + \sum_{j=1}^{\infty} |\epsilon_{ij}| \quad \}.$$

Let us defined $T$ by:

$$T(t) = e^{t(A+\Delta A)} \cos t(A + \Delta A) = e^{tA} \left(\frac{e^{it(A++\Delta A)} + e^{-it(A++\Delta A)}}{2}\right) =$$

$$\left(\frac{e^{(t+it)(A+\Delta A)} + e^{(t-it)(A+\Delta A)}}{2}\right) =$$

$$\frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(1+i)^n t(A+\Delta A)^n}{n!} + \sum_{n=0}^{\infty} \frac{(1-i)^n t(A+\Delta A)^n}{n!}\right) =$$

$$\frac{1}{2} \sum_{n=0}^{\infty} \frac{[(1+i)^n + (1-i)^n] t(A+\Delta A)^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^{\frac{n}{2}}}{n!}\; cos \frac{n\pi}{4} \left(t(A + \Delta A)\right)^n$$

(55)

such that

$$\|T(t)\|_\infty \le \frac{7}{4} e^{-t} \quad \forall\, t \ge 0. \tag{57}$$

Then, $A$ is clearly a generator of the semigroup $\{T(t)\}_{t \ge 0.}$

Furthermore, $0 \;\le t \le 1 \quad \Rightarrow 1 \le e^t \le e$ and we have:

$$\left\| f_1\big(t, x(t)\big) \int_0^t f_2\big(\tau, x(\tau)\big) d\tau \;\right\|_\infty \le \cos(\|x\|_\infty) \; \int_0^t \cos(\|x\|_\infty)\, d\tau$$

$$= \frac{1}{2} \cos(\|x\|_\infty)\, |\sin(\|x\|_\infty) - \sin(0)|$$

$\leq \frac{1}{2}|\cos(\|x\|_\infty) - \cos(0)||\sin(\|x\|_\infty) - \sin(0)|$

By Mean Value Theorem, there exists $m \in (0, \|x\|_\infty)$ such that:

$|\sin(\|x\|_\infty) - \sin(0)| \leq |\cos(m)|\|x\|_\infty$

$|\cos(\|x\|_\infty) - \cos(0)| \leq |\sin(m)|\|x\|_\infty$.

hence, $\left\| f_1\big(t, x(t)\big) \int_0^t f_2\big(\tau, x(\tau)\big) d\tau \right\|_\infty \leq \frac{1}{2}\|x\|_\infty$. (58)

for all $x, y \in l^\infty(\mathbb{N})$, we have that

$\left\| f_1\big(t, x(t)\big) \int_0^t f_2\big(\tau, x(\tau)\big) d\tau - f_1\big(t, y(t)\big) \int_0^t f_2\big(\tau, y(\tau)\big) d\tau \right\|_\infty \leq$

$\| f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, x(s)\big) ds -$

$f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, y(s)\big) ds +$

$f_1\big(\tau, x(\tau)\big) \int_0^\tau f_2\big(s, y(s)\big) ds - f_1\big(\tau, y(\tau)\big) \int_0^\tau f_2\big(s, y(s)\big) ds \| \leq$

$\frac{1}{2}\big(|cos(\|y\|_\infty)||\sin(\|x\|_\infty) - \sin(\|y\|_\infty)| +$

$|\cos(\|x\|_\infty) - \cos(\|y\|_\infty)|sin(\|y\|_\infty)\big)$

$\leq \frac{1}{2}(|\sin(\|x\|_\infty) - \sin(\|y\|_\infty)| + |\cos(\|x\|_\infty) - \cos(\|y\|_\infty)|)$

By Mean Value Theorem, there exists $r \in (\|x\|_\infty, \|y\|_\infty)$ such that

$|\cos(\|x\|_\infty) - \cos(\|y\|_\infty)| \leq |\sin(m)||\|x\|_\infty - \|y\|_\infty|$

$|\sin(\|x\|_\infty) - \sin(\|y\|_\infty)| \leq |\cos(m)||\|x\|_\infty - \|y\|_\infty|$

Hence,

$\|f(t,x) - f(t,y)\|_\infty \leq \frac{1}{2}\|x - y\|_\infty$ with $L = \frac{1}{2}$ (59)

We have the following:

$0 \leq t \leq 1 \Rightarrow -1 \leq -t \leq 0 \Rightarrow e^{-1} \leq e^{-t} \leq 1.$

So, from (57), we get, $\|T(t)\|_{\infty} \leq \frac{7}{4} = M$ (60)

and $\Omega = \frac{7}{2\times 4} \approx 0.875 < 1.$

then by Theorem (2.3.5) that the problem (51) has a unique mild solution on $[0,1]$. Furthermore, from (55) we get $\widetilde{\mathcal{M}} = \frac{7}{4}$ and $\tilde{\mu} = 1.$ hence,

$1 - \frac{7}{2\times 4} \approx 0.125 > 0.$ Hence, according to Theorem (2.1.7), the solution of problem (51) is exponentially stable.

**Conclusion**:

1. We concluded that the sufficient condition of linearization very interesting to satisfied on nonlinear integral function.
2. We concluded that some probability density function presented through processing of proving with stabilize control function supported the presented problems to be stabilization.
3. We concluded that to achieve a stability need a conditions depended on type of the system.
4. We concluded that all the examples are Clearfield the all classes are effective and can be used as a simulation mathematical of applied problems.

**References:**

[1] Al-Saphory, R., Khalid, Z. and El-Jai A. **2020.** Reginal boundary gradient closed loop control system and Γ*AGFO-observer. Journal of Physics: Conference Series, **1664** (012061): 1-19.

[2] A.M. Mathai and H.J. Haubold, An Introduction to Fractional Calculus, Mathematics Research Developments, Nova Science Publishers (2017).

[3] Li, H. L., Jiang, Y. L., Wang, Z. L., & Hu, C. (2015). Global stability problem for feedback control systems of impulsive fractional differential equations on networks. *Neurocomputing*, *161*, 155-161..

[4] Qasim Hasan, S., & Abbas Holel, M. (2018) Solution of Some Types for Composition Fractional Order Differential Equations Corresponding to Optimal Control Problems. Journal of Control Science and Engineering, volume 2018, Article ID 3767263, page1-12.

[5] Čermák, J., and Kisela, T. (2023). Stabilization and destabilization of fractional oscillators via a delayed feedback control. Communications in Nonlinear *Science and Numerical Simulation*, *117* (106960).

[6] Wang, X. (2018). Mittag-Leffler stabilization of fractional-order nonlinear systems with unknown control coefficients. *Advances in Difference Equations*, 1-14.

[7] Hasan S. Q. Stability for some types of multi control integro differential Inequalities (2022), *Journal of Interdisciplinary Mathematics* 26(4):659-674.

[8] Hasan S. Q. (2022). Uniform stability of integro-differential inequalities with nonlinear control inputs and delay. *Int. J. Nonlinear Anal. Appl*. 13 No. 1421 -430.

[9] Sattayatham Saelim R. and Sujitjorn S. *(2003)* Stability and stabilization of nonlinear dynamical system. *AJSTD Vol. 20 Issu*