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RESEARCH ARTICLE - MATHEMATICS

Some results on co fuzzy metric space

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Abstract

In this research, we presented the set G(X) which is the set for all fuzzy sets that are bounded functions, and we provided the definition of a cofuzzy metric, and we created some features the study of the two sets open and closed balls and also features fuzzy convergence and fuzzy closure set in the cofuzzy metric space.

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Keywords: fuzzy set, t co norm, co fuzzy metric space, open sphere, closed sphere.

1. Introduction

When zadeh first introduced the concept of Fuzzy set [1] in the year 1965 several authors have proposed the same concept as fuzzy metric spaces via various ways [2] Gorge and Vermani proposed that a fuzzy metric is induced by every metric [3]. They modified the concept of fuzzy metric space with the aid of t-norm and t-conorm in 1994. Many authors such as [4] [5] [6] [7] Provided properties representing the topological open set [8] The purpose is to clarify some properties of co fuzzy metric space throught the set G(X) and we created some features like the study of the two sets: open and closed balls and also features of co fuzzy convergence.

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2. Preliminaries

Definition 2.1. [9]

The binary operation (⊛): [0,l] × [0,l]→ [0,l] is a continuous t-norm if it the following requirements are satisfied:

(1) g ⊛ 1 = g

(2) g ⊛ h = h ⊗ g

(3) $g_1 \circledast h_1 \leq g_2 \circledast h_2 \quad for\ g_1 \leq g_2,\ h_1 \leq h_2$

(4) (g ⊛h) ⊛ k = g ⊛ (h ⊛ k)

Definition 2.2. [10]

The binary operation (⊗): [0,l] × [0,l]→ [0,l] is a continuous t-conorm if it the following requirements are satisfied:

(1) g ⊗ 0 = g

(2) g ⊗ h = h ⊗ g
(3) $g_1 \otimes h_1 \geq g_2 \otimes h_2 \quad for\ g_1 \geq g_2,\ h_1 \geq h_2$

(4) (g ⊗ h) ⊗ k = g ⊗ (h ⊗ k).

Example 2.1.

Consider (⊗): [0,l]× [0,l]→ [0,l] defined by g ⊗ h = max {g h}is t conorm.

Proof:

Let g, h, i, j ∈ [0, 1]

(1) g ⊗ 0 = max (g, 0) = g
(2) g ⊗ h = max (g, h) = max (h, g) = h ⊗ g
(3) g ≥ i , h ≥ j ⇒ max (g, h) ≥ max (i, j) ⇒ g ⊗ h ≥ i ⊗ j monutone
(4) (g ⊗ h) ⊗ i = max (g, h) ⊗ i = max ((g h) i) = max (g (h i)) = g ⊗ max(h i) = g ⊗ (h ⊗ i) assositive

Definition 2.3.[11]

The triple (X, N, ⊗) is called a co fuzzy metric space denoted by (Co- F M S) when X represents an arbitrary set, ⊗ is a continuous t-conorm and N is a fuzzy set of X × X × [0,∞) if it the following requirements are satisfied:

(1) N(g, h, 0) =1

(2) N(g, h, t) = 0, ∀ t > 0, ⇔ g = h

(3 N(g, h, t) = N(h, g, t), ∀ t > 0

(4) N(g, k, t + s)≤ N(g, h, t) ⊗ N(h, k, s)

(5) N(g, h, .) : X × X × [0,∞) ⟶ [0,1] is left continuous where g, h ∈ X and t, s > 0

given an (Co- F M S) N(g, h, t) we determined the open sphere $S_N(g,r,t)\ for\ g \in X, r \in (0,1)$

And t > 0 as the set $S_N(g,r,t) = \{h \in X : N\ (g,h,t) < r\}$

Obviously, g ∈ $S_N(g,r,t)$, ∀ g ∈ X, $0 < r_1 < r_2 < 1,\ and\ 0 < t_1 \leq t_2\ with\ us\ S_N(g,r_1,t_1) \subseteq S_N(g,r_2,t_2)$

Example 2.2.

Let $N(g,h,t) = (\frac{|g-h|}{t+|g-h|})$ ∀ g, $h \in X$, $t \in (0,\infty)$ and ⊗ is a continuous t-conorm then (X, N, ⊗) is (Co- F M S) .

Proof:

(1) $N(g,h,0) = \left(\frac{|g-h|}{t+|g-h|}\right) = \left(\frac{|g-h|}{|g-h|}\right) = 1,\ \forall\, t > 0$

(2) ∀ t > 0 Assume that g = h then this implies that | g - h| = 0

$N(g,h,t) = \left(\frac{|g-h|}{t+|g-h|}\right)\left(\frac{0}{t+0}\right) = 0$

(3) $N(g,h,t) = \left(\frac{|g-h|}{t+|g-h|}\right) = \left(\frac{[-(h-g)]}{t+[-(h-g)]}\right) = \left(\frac{|h-g|}{t+|h-g|}\right) = N(h,g,t).$

(4) $N\,(g,k,t+s) = \left(\frac{|g-h+h-k|}{(t+s)+|g-h+h-k|}\right) = \left(\frac{|g-h|}{t+|g-h|}\right) + \left(\frac{|h-k|}{s+|h-k|}\right).$

$\Longrightarrow\ N\,(g,k,t+s) \leq \left(\frac{|g-h|}{t+|g-h|}\right) + \left(\frac{|h-k|}{s+|h-k|}\right).$

$Then\ N\,(g,k,t+s) \leq N(g,h,t) \otimes N(h,k,s).$

(5) Take a sequence $\{t_n\} \in (0,\infty)$, $\ni$ $sequence\ \{t_n\}\ converge\ to\ t \in (0,\infty)\ where\ (0,\infty)$ equipped with the usual metric that is $|\,t_n - t\,| = 0$

Then every N(g, h, .) : (0, ∞) ⟶ [0, 1] is left continuous.

Definitions 2.4.

Suppose (X, N, ⊗) represent a (Co- F M S).

(1) $\lim_{n\to\infty} N\,(x_n, x, t) = 0$ then the sequence denote by (seq) in X is considered to be a convergent to a point . g ∈ X

(2) ∀ t > 0 further the seq $\{x_n\}$ in X is called a Cauchy seq in X if

$\lim_{n\to\infty} N\left(x_n, x_{n+p}, t\right) = 0,\ \forall\, t > 0\ and\ p > 0$

When each Cauchy seq of X converges to a point in X, then the space is considered to be complete.

Definition 2.5.

In (CO-FMS) the function N is continuous if whenever,

$\{x_n\} \to x, \{y_n\} \to y\ then\ \lim_{n\to\infty} N\,((x_n),(y_n),t) =, N(x,y,t)\ \forall\, t > 0$

Definition 2.6.

A mapping g from a (Co- F M S) (X, N, ⊗) to it self is continuous at x for all seq in X if $\lim_{n\to\infty} N\,((x_n),(x),t) = 0,\ t > 0,\ implies\ \lim_{n\to\infty} N\,(g(x_n), g(x), t) = 0$

3. Main Results

Let G(X) be a collection of all fuzzy sets in X, where X be a non-empty set, and
If g ∈ G(X) then g = {(x, α): x ∈ X and α ∈ (0, 1]

Obviously, f is a bounded function for | g (x) | ≤ 1 If K is the space of real numbers, therefore G(X) represents a space of vectors of a field K. The definitions of addition and scalar multiplication are as follows:

g + h = {(x, α): + (y, β)} = { x + y, α ⋀ β): (x, α) ∈ g, (y, β) ∈ h. And k g = {k (x, α): (x, α) ∈ g where k ∈ K}

a space of vectors G(X) is defined as a metric space if all g ∈ G(X), a function d: G(X) × G(X) → R is referred to as a metric function (distance function) on G(X) if it the following requirements are satisfied:

(1) d (g, h) ≥ 0 ∀ g, h ∈ G(X)
(2) d (g, h) = 0 iff f = h ∀ f ,h ∈ F(x)
(3) d (g, h) = d (h, f), ∀ f, h ∈ F(x)
(4) d (g, h) ≤ d (g, k) + d (k, h), ∀ g, h, k ∈ G(X), Then (G(X),d) is a metric space.

Dfinition 3.7.

Consider a linear space G(X) on the real field K. A (Co- F) sub set N of G(X) × G(X) → R is referred to as a (Co- F M) function on X (or co- F M) function on G(X) iff

(1) N(g, h, t) = 0, ∀ t ∈ R with t > 0
(2) N(g, h, 0) = 1 iff g and h linearly dependent, ∀ t ∈ R with t > 0
(3) N(g, h, t) = N(h, g, t),
(4) N(g, h, t + s) ≤ N(g, k, t) ⊗ N(k, h, s) ∀ t, s ∈ R
(5) N(g, h, .) (0, ∞) → [0,l] is left continuous
Then (G(X), N) is a (Co- F M S). ∀ g, h, k ∈ G(X)

Dfinition 3.8.

Let (G(X), N, ⊗) be a (Co- F M S)., and the open sphere is defined as S(g, r, t) with center g ∈ G(X) and radius r, 0 < r < l, t > 0, as S(g, r, t) = {g ∈ G(X) : N(g, h, t) < r }

Remark 3.1.

Let (G(X), N, ⊗) be a (Co- F M S)., and let g, h ∈ G(X), t > 0, 0 < r < l, Then if N(g, h, t) < r we can find t_0 $with\, 0 < t_0 < t \ni N(g, h, t_0) < r$

Theorem 3.1.

Let the open sphere $S(g, r_1, t)$ and $S(g, r_2, t)$ With a single center g ∈ G(X) and with radius $0 < r_1 < 1$ and $0 < r_2 < 1$ correspondingly. Then we any have
$S(g, r_1, t \subset S(g, r_2, t),\ or\ S(g, r_2, t) \subset S(g, r_1, t)$

Proof:

Let g ∈ G(X), t > 0 and consider $S(g, r_1, t)$ and $S(g, r_2, t)$ $with$ $0 < r_1 < 1$ and $0 < r_2 < 1$ are open sphere, $if\ r_1 = r_2$,then the hypothesis holds. Next, we suppose that $r_1 \neq r_2$. We may suppose without waste of energy. $If\ 0 < r_1 < r_2 < 1, Suppose\ r_1 < r_2\ Assuming\ an\ is\ in\ S(g, r_1, t)$, it follows that $N(a, h, t) < r_1 < r_2$. Assuming a belongs to $S(g, r_2, t)$, Attempts to demonstrate that $\mathrm{S(g, r_1, t)}$ is less than $\mathrm{S(g, r_2, t)}$. Assuming $0 < \mathrm{r_2} < \mathrm{r_1} < \mathrm{l}$, we may show that $\mathrm{S(g, r_2, t \subseteq S(g, r_1, t)}$.

Definition3.9.

Let A be a subset of the (Co- F M S). and let the collection (G(X), N, ⊗) considered open such that 0 < r < 1 if given any point a ∈ A and t > 0 then S(a, r, t) ⊆ A.

Theorem 3.2.

In a (Co- F M S). (G(X), N, ⊗), each open sphere represents an open set.

Proof:

Suppose an open sphere S(g, r, t), Now y ∈ S(x, r, t) infers that N(g, h, t) < r
Since N(g, h, t) < r by remark(3.l) it is possible to find a point t_0, $0 < t_0 < t \ni N(g, h, t) < r$

Let $r_0 = N(g, h, t) < r$ $since\, r_0 < r,\ to\, find\, s\, where, 0 < s < 1 \ni r_0 < s < r,$

Using a given r_0 and s in which $r_0 < s$ we can now find an s, $0 < s < 1 \ni N(g, h, t_0) < r$

Let $r_0 = N(g, h, t_0) since\, r_0 < r,$ it is possible to find an s, $0 < s < 1, \ni r_0 < s < r$

Now considering r_0 and s s.t $r_0 < s$, we can identify

$r_{1,}\ 0 < r_1 < 1\ , such\ that\ r_0 \otimes r_1 < s\ Now\ let\ the\ ball\ S(g, r_1, t - t_0), we\ say\ S(h, r_1, t - t_0) \subset S(g, r, t). Currently\ k \in S(h, r_1, t_0)\ involves\ that\ N(h, u, t - t_0) < r_1$

$$\begin{aligned} Therefore\ N(g, u, t) &< N(g, u, t_0) \otimes N(h, u, t - t_0) \\ &< r_0 \otimes r_1 \\ &< s \\ &< r. \end{aligned}$$

Hence u ∈ S (g, r, t), and therefore $S(h, r_1, t - t_0) \subset S(g, r, t)$.

Definition 3.10.

The (Co- F M S). (G(X), N, ⊗),we explain a closed sphere with the center g ∈ G(X) and the radius 0 < r < 1, t > 0, as S[g, r, t] = { g ∈ G(X) : N(g, h, t) ≤ r}

Lemma 3.1.

In a (Co- F M S). (G(X), N, ⊗), each closed sphere is a closed set.

Proof:

Since X is first countable, Let h ∈ S[g, r, t]. There is a seq
$\{h_n\}\ in\ \overline{S[g, r, t]}\ s.t\ the\ seq\{h_n\}\ \ convergence\ to\ h\ therefore\ N(h_n, h, t)$ converges to 0 for all t, for a given ϵ > 0

$$N(g, h, t + \varepsilon) < N(g, h_n, t) \otimes N(h_n, h, \varepsilon)$$

Hence

$$\begin{aligned} N(g, h, t + \varepsilon) &< \lim_{n\to\infty} N(g, h_n, t) \otimes N(h_n, h, \varepsilon) \\ &\leq r \otimes 0 \\ &= r \end{aligned}$$

$If\ N(\mathrm{g}, h_n, t)\ is\ bounded, the\ seq\ \{h_n\}$ contains a subseq denoted by $\{h_n\}\ for\ which\ \ lim_{n\to\infty} N(g, h_n, t)$ is existent, particularly for $n \in N$

$Take\ \ \varepsilon = \frac{1}{n}\ \ , then\ N(\mathrm{g}, h, t\ +\ \varepsilon\ )\ \ =\ lim_{n\to\infty} N(g, h, t + \frac{1}{n}\ )\ \ \leq\ \ r$

Thus g ∈ S [g, r, t] closed set

Definitions 3.11.

If the a (Co- F M S). (G(X), N, ⊗) , then

(a) The seq $\{g_n\}$ in G(X) is called (Co- F) convergent to x in G(X) if all $\epsilon \in (0,1)$ and all t > 0

$\exists\ n_0\ \in Z^+\ \ni\ N(g_n, g_m, t)\ <\ \varepsilon\ \ \forall\ n\ \geq n_0$

(b) The seq $\{g_n\}$ in X is called (Co- F) Cauchy seq if for every x in G(X) if $\forall\, \epsilon \in (0,1)$ and each t > 0
$\exists\ n_0\ \in Z^+\ \ni\ N(g_n, g_m, t)\ <\ \varepsilon\ \ \forall\ n, m\ \geq n_0$

(c) A (Co- F M S) s.t every (Co- F) Cauchy seq is (Co- F) convergent is referred to as complete

Theorem 3.3.

(i) Each fuzzy convergent seq is (Co- F) Cauchy seq in (Co- F M S). (G(X), N, ⊗)

(ii) In G(X), each seq has an unique limit.

Proof:

(i) Suppose that $\{g_n\}$ is the seq in G(X) ∋ , ∀ t > s > 0

$\lim_{n\to\infty} N(g_n, g, t\ ) = 0$

$N(g_n, g, t\ ) \leq\ N(g_n, \mathrm{g}, t\ -\ s)\ \otimes\ N(g_m, \mathrm{g}, s)$ Taking limit as m, n ⟶ ∞

$\lim_{n\to\infty} N(g_n, g_m, t\ ) \leq\ lim_{n\to\infty} N(g_n, g, t - s\ ) \otimes lim_{n\to\infty} N(g_m, g, s\ ) =\ 0\ \otimes\ 0\ =\ 0$

$But\ \ lim_{n,m\to\infty} N(g_n, g_m, t\ ) \leq\ 0\ \ then\ \ lim_{n,m\to\infty} N(g_n, g_m, t\ ) = 0\ ,\ \Rightarrow \{g_n\}$ is (Co- F) Cauchy seq in X .

(ii) $Let\ \{g_n\} is\ the\ seq\ in\ G(X)\ \ni\ \ g_n \longrightarrow\ \ \mathrm{g}\ and\ g_n \longrightarrow\ \ h\ and\ \ \mathrm{g}\ \neq\ h\ then\ \forall\ t\ >\ s\ >\ 0 >$
$lim_{n\to\infty} N(g_n, g, t\ ) = 0$

$Then\ \ lim_{n\to\infty} N(g_n, h, s\ ) = 0,\ \ \ lim_{n\to\infty} N(g_n, h, t - s\ ) = 0$

$N(\mathrm{g}, h, t)\ \leq\ N(g_n, \mathrm{g}, s)\ \otimes\ \ N(g_n, h, t\ -\ s)$

Taking limit

$N(\mathrm{g}, h, t)\ \leq\ lim_{n\to\infty} N(g_n, g, s\ )\ \otimes\ lim_{n\to\infty} N(g_n, g, t - s\ ) = 0$

N(g, h, t) $\leq$ 0 $\otimes$ 0 = 0 but N(g, h, t) $\leq 0 \Rightarrow$ N(g, h, t) = 0

Then by axiom (2) g = h

Definition 3.12.

If (G(X), N, $\otimes$) is a (Co- F M S)., then the (Co- F) closure of A is defined as a subset $\bar{A}$ of G(X)

$A \subset G(X)\ if\ for\ any$ g $\in \bar{A},\ a \in G(X)$ there exists a seq
$\{g_n\}\ in\ A \ \ni\ lim_{n\to\infty} N(g_n, g, t) = 0,\ \forall\ t > 0$

Theorem 3.4.

Let A be a (Co- F) subspace of complete (Co- F M S). G(X) then A is complete (Co- F) space iff it is (Co- F) closed in G(X)

Proof

Let A be a complete (Co- F M S). and let $g \in \bar{A}$ there exist a seq $\{g_n\}\ in\ A \ni\ g_n \to g\ then\ \{g_n\}$
Is a (Co- F) Cushy seq in A, since A is a complete (Co- F) space.

$\Longrightarrow\ there\ is\ h \in \bar{A} \ni g_n \longrightarrow h,$ but the (Co- F) converge is unique

$h \Longrightarrow\ g \Longrightarrow\ g \in A \Longrightarrow\ \bar{A} \subseteq\ A$

Then A is closed (Co- F) subspace.

Conversely, let us assume that A is a closed (Co- F) subspace within G(X)

$Let\ \{g_n\}$ be a (Co- F) Cauchy seq in A

$Since\ A \subset G(X) \Longrightarrow \{g_n\}$ is a (Co- F) Cauchy seq in G(X)

Because G(X) is a complete fuzzy space, there is $g \in G(X) \ni\ g_n \longrightarrow\ g\ since\ g_n\ \in A \Longrightarrow\ g \in\ \bar{A}$

Since A is closed (Co- F) set in G(X),

$\bar{A} =\ A \Longrightarrow\ g \in A \Longrightarrow \{g_n\}$ is (Co- F) converge seq in A then A is complete (Co- F) subspace

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