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RESEARCH ARTICLE - MATHEMATICS

An Information-Theoretic Approach for Multivariate Skew Laplace Normal Distributions

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Abstract

Due to its flexibility, the skew distributions (univariate and multivariate) have received widespread attention over the last two decades because they're become widely used in the modelling and analysis of skewed data sets. The main goal of this paper is to introduce asymptotic expressions for entropy of multivariate skew Laplace normal distribution to deal with the issue by providing a flexible model for modeling skewness and heavy tiredness simultaneously. Thus, we extend this study to the class of mixture model of these distributions. In addition, upper and lower bounds of Rényi entropy of mixture model are found, by using generalized HÖlder's inequality and some properties of multinomial theorem.. Finally, we give a real data examples to illustrate the behavior of information. A simulation study and a real data example, are also provided to illustrate the information behavior of MSLN and MMSLN distributions for modeling data sets in multivariate settings.

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***Keywords**:* Rényi entropy, mixture model, MSLN, MMSLN and multinomial theorem.

1. Introduction

The mixture models are an important statistical tool for many applications such as data mining, density estimation, pattern recognition, medicine, satellite imaging and image processing etc. (More detail [1], [2], [3], [4]).

In 1996 Azzalini[5] proposed the skew normal distribution as an alternative to normal distribution. After a short period of time, Genton and Loperfido (2005) [6] introduced the generalization of multivariate skew normal distribution as follows

$$f_d(y;\xi,S) = 2\varphi(y;\xi,S).\,Ꝁ(y-\xi)\ , \quad y \in R^d \tag{1}$$

Where, $\varphi(y;\xi,S)$ is a probability density function of multivariate normal distribution and $0 \le Ꝁ(y) \le 1$ such that $Ꝁ(-y) = 1 - Ꝁ(y)$, for any $y \in R^d$. Clearly, if we take $Ꝁ(y) = \frac{1}{2}$ Then Y has a normal distribution in the multivariate case. Where, if Ꝁ is a distribution function such as $Ꝁ(y) = G(\eta' y)$ Then this gives the generalized skew normal distribution. The structure of this paper was designed according to the following fundamental aims:

1) We determine and investigate the exact expression for Shannon and Rènyi entropies of the skew Laplace normal distributions in multivariate analysis.
2) We drive and identify the best approximate expression for Shannon entropy of mixture model of distributions.

3) An asymptotic expression for Rényi entropy is given by the approximation and by using some inequalities and properties of Lp-spaces

4) We find the bounds of entropy for mixture model of distributions by using generalized HÖlder's inequality and some properties of multinomial theorem.

5) We give and discuss examples with real data to illustrate the behavior of entropy of the mixture models.

2. Preliminary Material

Let y be a continuous random vector that takes values of R^d and $\mathcal{P}(y;\xi)$be the density function of y. A continuous Rényi entropy of y is defined as

$$H_{\boldsymbol{\tau}}(y;\xi)=\begin{cases}\dfrac{1}{1-\boldsymbol{\tau}}\ln(E(\mathcal{P}(y;\xi)\,)^{\boldsymbol{\tau}-1})\ , \boldsymbol{\tau}\neq 1\,, 0<\boldsymbol{\tau}, \\ -E\left(\ln\left(\mathcal{P}(y;\xi)\right)\right) \qquad , \qquad \boldsymbol{\tau}=1\end{cases} \tag{2}$$

When $\tau=1$, we have the Shannon entropy

$$H_1(y;\xi)=-E\left(\ln\left(\mathcal{P}(y;\xi)\right)\right) \tag{3}$$

The relationship between the entropies in equations **(1)** and **(2)** is given by $H_1(y;\xi)=\lim_{\boldsymbol{\tau}\to 1}H_{\boldsymbol{\tau}}(y;\xi)$. Translation does not change of the entropy $H_{\boldsymbol{\tau}}(y+k;\xi)=H_{\boldsymbol{\tau}}(y;\xi)+k$ where, k is constant.

Proposition 1. Suppose that $0<\boldsymbol{\tau}_1<\boldsymbol{\tau}_2<\infty$, then $H_{\boldsymbol{\tau}_1}(y;\xi)\geq H_{\boldsymbol{\tau}_2}(y;\xi)$. If y has uniformly distribution then $H_{\boldsymbol{\tau}_1}(y;\xi)=H_{\boldsymbol{\tau}_2}(y;\xi)$.

Proof: firstly**,** suppose that $\boldsymbol{\tau}\neq 1$, then

$$\frac{\partial}{\partial\boldsymbol{\tau}}H_{\boldsymbol{\tau}}(y;\xi)=\frac{E\left(\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\ln\left(\mathcal{P}(y;\xi)\right)\right)}{(1-\boldsymbol{\tau})E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}+\frac{\ln\left(E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\right)}{(1-\boldsymbol{\tau})^2}$$

The second part in right side of the above equation is

$$\frac{\ln\left(E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\right)}{(1-\boldsymbol{\tau})^2}=\frac{E\left(\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\ln\left(E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\right)\right)}{(1-\boldsymbol{\tau})^2E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}$$

Therefore,

$$\frac{\partial}{\partial\boldsymbol{\tau}}H_{\boldsymbol{\tau}}(y;\xi)=\frac{1}{(1-\boldsymbol{\tau})^2}\left\{\frac{E\left(\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\ln\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\right)}{E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}+\frac{E\left(\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\ln\left(E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\right)\right)}{E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}\right\}$$

$$=\frac{-1}{(1-\boldsymbol{\tau})^2}\left\{\frac{E\left(\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}\ln\left(\dfrac{\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}{E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}\right)\right)}{E\left(\mathcal{P}(y;\xi)\right)^{\boldsymbol{\tau}-1}}\right\}$$

Now, to define $g(y;\xi)=\frac{(\mathcal{P}(y;\xi))^{\boldsymbol{\tau}}}{E(\mathcal{P}(y;\xi))^{\boldsymbol{\tau}-1}}$ As a new probability function, then

$$\frac{\partial}{\partial\boldsymbol{\tau}}H_{\boldsymbol{\tau}}(y;\xi)=\frac{-1}{(1-\boldsymbol{\tau})^2}\left\{E_g\left(\ln\left(\frac{g(y;\xi)}{\mathcal{P}(y;\xi)}\right)\right)\right\}$$

But $E_g\left(\ln\left(\frac{g(y;\xi)}{\mathcal{P}(y;\xi)}\right)\right)$ not negative value therefor, either $E_g\left(\ln\left(\frac{g(y;\xi)}{\mathcal{P}(y;\xi)}\right)\right)>0$, then $H_{\boldsymbol{\tau}}(y;\xi)$ is decreasing with respect to $\boldsymbol{\tau}$ or $E_g\left(\ln\left(\frac{g(y;\xi)}{\mathcal{P}(y;\xi)}\right)\right)=0$ this implies that $f=g$ almost everywhere, but $\boldsymbol{\tau}\neq 1$ then $\mathcal{P}$ only has uniform distribution. Conversely, assume that y has uniform distribution then, $\mathcal{P}(y;\xi)=\begin{cases}\frac{1}{\mathrm{Vol}(A)} & ,\ x\in A\\ 0 & ,x\notin A\end{cases}$, for some measurable set $A\subseteq R^n$. consequently, $H_{\boldsymbol{\tau}}(y;\xi)=\frac{1}{1-\boldsymbol{\tau}}\ln(\mathrm{Vol}(A)^{\boldsymbol{\tau}-1})=\ln(\mathrm{Vol}(A))$ Does not depend on $\boldsymbol{\tau}$.

Proposition 2. Rènyi entropy does not depend on the location parameter for location scale model does not depend on location parameter.

Proof: The location scale model $\mathcal{P}(y;\xi)$ Can be written as

$$\mathcal{P}_y(y;\xi,\Omega)=\left(\det(\Omega)\right)^{-\frac{1}{2}}\mathcal{P}_{y_0}\left(\Omega^{-\frac{1}{2}}(y-\xi),0,I\right)$$

Where, $y_0=\Omega^{-\frac{1}{2}}(y-\xi)$. The Rènyi entropy is

$$H_{\boldsymbol{\tau}}(y;\xi,\Omega)=\ln\left(\det(\Omega)\right)^{\frac{1}{2}}+\frac{1}{1-\boldsymbol{\tau}}\ln\left(E\ \left(\mathcal{P}_{y_0}(y_0,0,I)\right)\right)^{\boldsymbol{\tau}-1}$$
$$=\ln\left(\det(\Omega)\right)^{\frac{1}{2}}+H_{\boldsymbol{\tau}}(y;0,1)$$

Proposition 3. [7] Let $y_0\sim MN_d(\xi,\Omega)$. Then, the Rényi entropy of y_0 is given as

$$H_{\boldsymbol{\tau}}(y;\xi,\Omega)=\begin{cases}\frac{1}{2}\ln\left(\det(2\pi\exp(1)\,\Omega)\right) & ,\ \boldsymbol{\tau}=1\\ \frac{1}{2}\ln\left(\det(2\pi\Omega)\right)-\frac{d}{2(1-\boldsymbol{\tau})}\ln(\boldsymbol{\tau}) & ,0<\boldsymbol{\tau}<\infty,\boldsymbol{\tau}\neq 1\end{cases} \tag{4}$$

Lemma 1. [8] Let y be a continuous random vector that takes values of R^d and $\mathcal{P}(y;0,\mathcal{M})$ be the density function of y. Then, the following inequality is accomplished

$$H_1(y;0,\mathcal{M})\leq\frac{1}{2}\ln\left(\det(2\pi\exp\,(1)\mathcal{M})\right) \tag{5}$$

3. Entropy of proposed Distributions

This section includes the complete derivation of simple expressions for entropy of multivariate skew Laplace normal distribution. Some properties of transformations and integrations are used. Also, we give an illustrative example explains the relationship between the parameter **τ**th order of entropy and skewness parameter η with the values of Rènyi entropy.

Azzalini and Capitaino [9] introduced multivariate skew Laplace distribution($\mathrm{y} \sim \mathrm{MSL_d}(\xi, \Omega, \eta)$) as follows

$$\mathcal{P}_{\mathrm{d}}(\mathrm{y}; \xi, \Omega, \eta) = \frac{\det(\Omega)^{-\frac{1}{2}}}{2^{\mathrm{d}}\beta(\pi)^{\frac{1}{2}(\mathrm{d}-1)}\Gamma\left(\frac{\mathrm{d}+1}{2}\right)} \exp\left\{ \begin{array}{c} -\sqrt{(1+\eta'\Omega^{-1}\eta)(\mathrm{y}-\xi)'\Omega^{-1}(\mathrm{y}-\xi)} \\ +(\mathrm{y}-\xi)'\Omega^{-1}\eta \end{array} \right\} \tag{6}$$

Where, $\xi \in \mathrm{R^d}$, $\Omega \in \mathrm{R^{d \times d}}$ and skewness vector $\eta \in \mathrm{R^d}$.

The generalized skew Laplace distribution ($\mathrm{y} \sim \mathrm{SGL}(\xi, \Omega, \eta_1, \eta_2)$) is

$$\mathrm{k}(\mathrm{y}; \xi, \Omega, \eta_1, \eta_2) = 2\mathrm{g}(\mathrm{y}; \xi, \Omega).\,\mathrm{G}\left(\frac{\eta_1 y}{(1+\eta_2\, y^2)^{\frac{1}{2}}}\right) \tag{7}$$

where, g is univariate Laplace distribution, G is cumulative distribution function of univariate standard normal and $\eta_1 \in \mathrm{R}$, $\eta_2 \geq 0$. A random vector $\mathrm{y} \in \mathrm{R^d}$ has multivariate skew Laplace normal distribution ($\mathrm{y} \sim \mathrm{MSLN_d}(\xi, \Omega, \eta)$) if it has probability density function as follows

$$\psi_d(\mathrm{y}; \xi, \Omega, \eta) = 2\mathcal{P}_{\mathrm{ML}}(\mathrm{y}; \xi, \Omega)\Phi\left(\eta'\widetilde{\Omega}^{-1}(\mathrm{y}-\xi)\right) \tag{8}$$

where, $\xi, \eta \in \mathrm{R^d}$, $\Omega \in \mathrm{R^{d \times d}}$ (positive definite), $\widetilde{\Omega} = \mathrm{diag}(\mathrm{s}_{11}, \mathrm{s}_{22}, \dots, \mathrm{s}_{\mathrm{dd}})^{\frac{1}{2}}$, $\Omega = (\mathrm{s_{ij}})$, i ,j =1,2,…,d and $\mathcal{P}_{\mathrm{ML}}(\mathrm{y}; \xi, \Omega)$ is a multivariate Laplace distribution.

$$\mathcal{P}_{\mathrm{ML}}(\mathrm{y}; \xi, \Omega) = \frac{\det(\Omega)^{-\frac{1}{2}}}{2^{\mathrm{d}}(\pi)^{\frac{1}{2}(\mathrm{d}-1)}\Gamma\left(\frac{\mathrm{d}+1}{2}\right)} \exp\left\{-\sqrt{(\mathrm{y}-\xi)'\Omega^{-1}(\mathrm{y}-\xi)}\right\} \tag{9}$$

The stochastic representation of $y \sim \mathrm{MSLN_d}(\xi, \Omega, \eta)$ can be introduced as mixture of multivariate normal distribution $\mathrm{U_1} \sim \mathrm{MN_d}(\xi, \Omega)$ and univariate standard normal distribution $\mathrm{U_2} \sim \mathrm{N}(0,1)$

$$y = \xi + (\Omega)^{\frac{1}{2}}\left[\frac{\eta|\mathrm{U_2}|}{\sqrt{\upsilon(\upsilon+\eta'\eta)}} + (\upsilon I_d + \eta\eta')^{\frac{1}{2}}\mathrm{U_1}\right] \tag{10}$$

Where, υ has the inverse gamma distribution with the probability density function

$$g(\upsilon) = \frac{1}{2^{\frac{\mathrm{d}+1}{2}}\Gamma\left(\frac{\mathrm{d}+1}{2}\right)}\, \upsilon^{-\left(\frac{\mathrm{d}+1}{2}+1\right)} e^{-\frac{1}{\upsilon}} \tag{11}$$

and suppose that U_1, U_2 and υ are mutually independent. For the details about this see [10].

The joint density function and conditional distribution are given as follows ([11])

$$\mathcal{P}(x,z) = \frac{\det(\Omega)^{-\frac{1}{2}} \exp\{(x-\xi)'\Omega^{-1}(x-\xi)\}}{2^{d}\sqrt{(1+\eta'\Omega^{-1}\eta)}(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)} z^{-\frac{3}{2}}$$
$$. \exp\left\{-\frac{1}{2}((x-\xi)'\Omega^{-1}(x-\xi)z + (1+\eta'\Omega^{-1}\eta)z^{-1})\right\} \tag{12}$$

$$\mathcal{P}(z|x) = \frac{\sqrt{(1+\eta'\Omega^{-1}\eta)}}{\sqrt{2\pi}} \exp\left\{ \begin{array}{l} -\sqrt{(1+\eta'\Omega^{-1}\eta)}z^{-\frac{3}{2}}\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)} \\ -\frac{1}{2}((x-\xi)'\Omega^{-1}(x-\xi)z + (1+\eta'\Omega^{-1}\eta)z^{-1}) \end{array} \right. \tag{13}$$

Clearly, when $\eta = 0$ then the multivariate skew Laplace distribution can be reduced to the symmetric multivariate Laplace distribution. The mean vector and covariance matrix of $y$are derived by [10] in the following forms:

$$E(y) = \xi + \sqrt{\frac{2}{\pi}}(\Omega)^{\frac{1}{2}}\eta\,\Upsilon \tag{14}$$

$$\mathrm{Var}(y) = (d+1)\Omega - \frac{2}{\pi}(\Omega)^{\frac{1}{2}}\,\eta\eta'(\Omega)^{\frac{1}{2}}\theta^2 \tag{15}$$

where, $\Upsilon = E\left(\frac{(\upsilon)^{-\frac{1}{2}}}{\sqrt{\upsilon+\eta'\eta}}\right)$, , which can be computed by using numerical methods such as importance sampling methods.

The characteristic function of $y \sim MSLN_d(\xi,\Omega,\eta)$ is given by

$$\Psi_X(r) = \exp\left(\frac{ir'\xi}{2}\right) E_\upsilon\left(\exp\left(\frac{-\upsilon^{-1}r'\Omega r}{2}\right)(1+i\tau\kappa'\omega r)\right) \tag{16}$$

where, $\kappa = \frac{\upsilon^{-\frac{3}{2}}\tilde{\Omega}\eta}{(1+\upsilon^{-2}\eta'\tilde{\Omega}\eta)^{\frac{1}{2}}}$, $\tilde{\Omega} = \omega\Omega\omega$ and $\omega = \mathrm{diag}(s_{11}, s_{22}, \ldots, s_{dd})^{\frac{1}{2}}$, $\Omega = (s_{ij})$, $i, j = 1,2,\ldots,d$ and the function $\tau(\cdot)$ is defined as follows

$$\tau(y) = \int_0^y \sqrt{\frac{2}{\pi}}\, exp\left(\frac{u^2}{2}\right) du \quad , \qquad y > 0\,, \tau(-y) = -\tau(y)$$

We note that the expectation given in the equation (**16**) can be calculated by numerical methods.

Proposition 4. Let $X_0 \sim ML_d(\xi,\Omega)$. Then

$$H_1(x_0;\xi,\Omega) = -2\ln\left(\frac{\det(\Omega)^{-\frac{1}{2}}}{2^d(1+\eta'\Omega^{-1}\eta)(\pi)^{\frac{1}{2}(d-1)}\,\Gamma\left(\frac{d+1}{2}\right)}\right) - 2d \tag{17}$$

$$H_{\boldsymbol{\tau}}(x_0;\xi,\Omega) = \ln\left[\frac{\boldsymbol{\tau}^{-\frac{d}{1-\boldsymbol{\tau}}}\det(4\pi\Omega)^{\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\right], 0<\boldsymbol{\tau}<\infty, \boldsymbol{\tau}\neq 1 \tag{18}$$

Proof: By taking the natural logarithm and expectation for both sides of equation (**9**), we get on equation (**17**)**.** Also, from equation (**9**), we get

$$\int_{R^d}\left(\mathcal{P}_{ML}(y;\xi,\Omega)\right)^{\boldsymbol{\tau}} dx = \frac{\det(\Omega)^{-\frac{1}{2}\boldsymbol{\tau}}}{2^{\boldsymbol{\tau}(d-1)}(\pi)^{\frac{1}{2}\boldsymbol{\tau}(d-1)}\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{\boldsymbol{\tau}}}\int_{R^d}\exp\left\{-\boldsymbol{\tau}\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right\} dx$$

Taking natural logarithm and multiplying by $1-\boldsymbol{\tau}$ for both sides of above equation, we have the result in equation (18)

Lemma 2. Let $X \sim MSLN_d(\xi,\Omega,\eta)$ distributed in the equation (**8**). Then

i. $$E\left[ln\left(\Phi\left(\eta'\widetilde{\Omega}^{-1}(x-\xi)\right)\right)\right] = \sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon \tag{19}$$

ii. $$E\left[\left(\ln\left(\mathcal{P}_{ML}(x;\xi,\Omega)\right)\right)\right] = \ln\left(\frac{\det(\Omega)^{-\frac{1}{2}}}{2^d\sqrt{(1+\eta'\Omega^{-1}\eta)}(\pi)^{\frac{1}{2}(d-1)}\,\Gamma\left(\frac{d+1}{2}\right)}\right) + (d+1)\eta'\Omega^{-1}\eta + d \tag{20}$$

Proof: Directly,

$$E\left[ln\left(\Phi\left(\eta'\widetilde{\Omega}^{-1}(x-\xi)\right)\right)\right] = E[(x-\xi)'\Omega^{-1}\eta]$$

$$= E\left(Tr\left(\eta'\Omega^{-1}(x-\xi)\right)\right)$$

$$= Tr\left(\eta'\Omega^{-1}E(x-\xi)\right)$$

$$= Tr\left(\sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon\right)$$

$$= \sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon$$

Now, to prove part ii.,

$$E\left[\left(\ln\left(\mathcal{P}_{ML}(x;\xi,\Omega)\right)\right)\right] = ln\left(\frac{\det(\Omega)^{-\frac{1}{2}}}{2^d(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right) + E\left\{\ln\left(-\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right)\right\}$$

if we use the equation (**13**) then the conditional expectation of Z^{-1} given X is

$$E(z^{-1}|x) = \frac{1}{(1+\eta'\Omega^{-1}\eta)}\left(1-\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right)$$

where, $Z^{-1} \sim G(\frac{d+1}{2}, \frac{1}{2})$. Taking expectation for both sides of the above equation, we have

$$E\big(E(z^{-1}|x)\big) = \frac{1}{(1+\eta'\Omega^{-1}\eta)}\left(1+E\left[-\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right]\right)$$

Therefore,

$$E(z^{-1}) = \frac{1}{(1+\eta'\Omega^{-1}\eta)}\left(1+E\left[-\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right]\right)$$

But $Z^{-1} \sim G(\frac{d+1}{2}, \frac{1}{2})$ then

$$d+1 = \frac{1}{(1+\eta'\Omega^{-1}\eta)}\left(1+E\left[-\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right]\right)$$

Consequently,

$$E\left[\Big(\ln\big(\mathcal{P}_{ML}(x;\xi,\Omega)\big)\Big)\right] = ln\left(\frac{\det(\Omega)^{-\frac{1}{2}}}{2^{d}(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right) + (d+1)\eta'\Omega^{-1}\eta + d$$

Proposition 5. Let $X_0 \sim ML_d(\xi,\Omega)$ and $X \sim MSLN_d(\xi,\Omega,\eta)$. Then

$$H_1(X;\xi,\Omega,\eta) = H(X_0;\xi,\Omega) - \hat{C}_{\tilde{\eta}} \tag{21}$$

where,

$$\hat{C}_{\tilde{\eta}} = 2(d+1)\eta'\Omega^{-1}\eta + \sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon \tag{22}$$

Proof: from **equation** (**8**), we have

$$H_1(X;\xi,\Omega,\eta) = -2E\Big(\ln\big(\mathcal{P}_{ML}(x;\xi,\Omega)\big)\Big) - E\left(\ln\left(\Phi\left(\eta'\tilde{\Omega}^{-1}(x-\xi)\right)\right)\right)$$

Using lemma 2., we obtain

$$H_1(X;\xi,\Omega,\eta) = -2\ln\left(\frac{\det(\Omega)^{-\frac{1}{2}}}{2^{d}\sqrt{(1+\eta'\Omega^{-1}\eta)}(\pi)^{\frac{1}{2}(d-1)}\Gamma\left(\frac{d+1}{2}\right)}\right) - 2d - 2(d+1)\eta'\Omega^{-1}\eta - \sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon$$

From Proposition 4., we have

$$H_1(X;\xi,\Omega,\eta) = H_1(X_0;\xi,\Omega) - 2(d+1)\eta'\Omega^{-1}\eta - \sqrt{\frac{2}{\pi}}\eta'\Omega^{-\frac{1}{2}}\eta\Upsilon$$

Lemma 3. Let $X \sim MSLN_d(\xi,\Omega,\eta)$. Then:

$$\int_{R^d} \left(\psi_d(x;\xi,\Omega,\eta)\right)^{\tau} dx = \left(\frac{\tau^{-\frac{d}{1-\tau}}\ \det(4\pi\Omega)^{\frac{1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)\sqrt{(1+\eta'\Omega^{-1}\eta)}}{\sqrt{\pi}}\right)^{(1-\tau)} \qquad \textbf{(23)}$$

Proof: from equation (**8**), we get

$$\int_{R^d} \left(\psi_d(x;\xi,\Omega,\eta)\right)^{\tau} dx = \int_{R^d} \frac{\det(\Omega)^{-\frac{1}{2}\tau}}{2^{\tau(d-1)}(\pi)^{\frac{1}{2}\tau(d-1)}\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{\tau}} \exp\left\{-\tau\sqrt{(x-\xi)'\Omega^{-1}(x-\xi)}\right\}\left(\Phi\left(\eta'\tilde{\Omega}^{-1}(x-\xi)\right)\right)^{\tau} dx$$

Replacing $\Omega^{-\frac{1}{2}}\eta$ by $\tilde{\eta}$, $\tau^{-2}\Omega$ by Ω_{τ} and using the change of variables $Y=\Omega_{\tau}^{-\frac{1}{2}}(X-\xi)$ associated with Jacobian matrix $\Omega^{\frac{1}{2}}$ In the above equation, we get

$$\int_{R^d} \left(\psi_d(x;\xi,\Omega,\eta)\right)^{\tau} dx = \frac{\tau^{-\tau d}\det(\Omega_{\tau})^{-\frac{1}{2}\tau+\frac{1}{2}}}{2^{\tau d}(\pi)^{\frac{1}{2}\tau(d-1)}\left(\Gamma\left(\frac{d+1}{2}\right)\right)^{\tau}} \int_{R^d} \exp\left\{-\sqrt{y'}y+y'\tilde{\eta}\right\} dy$$

where, $Y\sim ML_d(0,I_d,\tilde{\eta})$. This complete the proof.

Corollary 1. If $X_0\sim ML_d(\xi,\Omega)$ and $X\sim MSLN_d(\xi,\Omega,\eta)$., then the Rényi entropy can be written as

$$H_{\tau}(X;(\xi,\Omega,\eta) = H_{\tau}(X_0;(\xi,\Omega,\eta) + Ꝃ_{\eta} \qquad (24)$$

Where, $Ꝃ_{\eta} = \ln\left(\sqrt{(1+\eta'\Omega^{-1}\eta)}\right)$

Proof : Multiplying by $\frac{1}{1-\tau}$ for equation (**23**), we have

$$H_{\tau}(X;(\xi,\Omega,\eta) = \frac{1}{1-\tau} ln\left(\frac{\tau^{-\frac{d}{1-\tau}}\ \det(4\pi\Omega)^{\frac{1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)\sqrt{(1+\eta'\Omega^{-1}\eta)}}{\sqrt{\pi}}\right)^{(1-\tau)}$$

Hence,

$$H_{\tau}(X;(\xi,\Omega,\eta) = ln\left(\frac{\tau^{-\frac{d}{1-\tau}}\ \det(4\pi\Omega)^{\frac{1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\right) + ln(1+\eta'\Omega^{-1}\eta)$$

Proposition 4. gives us the result of this corollary.

4. Computational Implementation and Numerical Simulations

All numerical computations were made with MATLAB 2020a. The integrals of skew Laplace normal Shannon entropy of Equation (**21**)were evaluated using numerical methods such as Monte Carlo and importance sampling methods. This section illustrates the relationship between parameters $\boldsymbol{\tau}$ and η with entropy for d = 1, 2, 3, and 4 dimensions. Consider $\mathrm{X}{\sim}\mathrm{MSLN}_{\mathrm{d}}(\xi,\Omega,\eta)$ with the following parameters:

Case (1) d=1, $\xi = 0.9,\ \Omega = 0.5,\ \eta = 0.5$

Case (2) d=2, $\xi = \begin{pmatrix}1\\2\end{pmatrix}, \Omega = \begin{pmatrix}0.7 & 0.1\\0.1 & 1\end{pmatrix}, \eta = \begin{pmatrix}0.5\\1\end{pmatrix}$

Case(3) d=3, $\xi = \begin{pmatrix}0\\1\\0\end{pmatrix}, \Omega = \mathrm{I}_3, \eta = \begin{pmatrix}0.5\\1\\0.4\end{pmatrix}$

Case(4) d=3, $\xi = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \Omega = \mathrm{I}_4, \eta = \begin{pmatrix}0.5\\1\\0.4\\0.1\end{pmatrix}$

Case(5) d=3, $\xi = \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \Omega = \mathrm{I}_5, \eta = \begin{pmatrix}0.5\\1\\0.4\\0.1\\1\end{pmatrix}$

Table 1. The values of entropy of $\mathrm{MSLN}_{\mathrm{d}}(\xi,\Omega,\eta)$ are computed for $\boldsymbol{\tau}$ = 2,3,4,5,10,20 and $\boldsymbol{\tau}$ converges to infinite of one to five dimensions

Case	*Shannon entropy* $\mathrm{H}_{\boldsymbol{\tau}}(\mathrm{x};\xi,\Omega,\eta)$						
d	$\mathrm{H}(\mathrm{x};\xi,\Omega$	$\boldsymbol{\tau}=2$	$\boldsymbol{\tau}=3$	$\boldsymbol{\tau}=4$	$\boldsymbol{\tau}=5$	$\boldsymbol{\tau}=10$	$\boldsymbol{\tau}=20$	$\boldsymbol{\tau}\to\infty$
1	1.9250	1.6182	1.4743	1.3871	1.3274	1.1809	1.0827	1.0423
2	4.6138	4.0001	3.7124	3.5380	3.4185	3.1255	2.9291	2.8483
3	6.7775	5.8569	5.4254	5.1638	4.9845	4.5450	4.2505	4.1293
4	9.5977	8.3703	7.7950	7.4461	7.2072	6.6211	6.2284	5.6254
5	12.359	10.825	10.106	9.6703	9.3716	8.6390	8.1482	7.3604

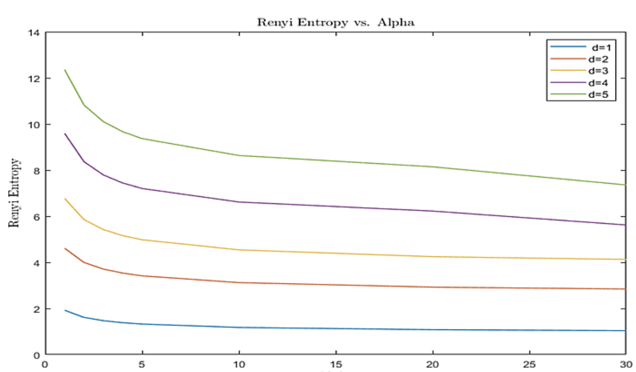

Fig. 1. The horizontal line represents the values of parameter α and the vertical line is the Rényi entropy of $\mathrm{X}{\sim}\mathrm{MSL}_{\mathrm{d}}(\xi,\Omega,\eta)$ with parameters in example 1.

5. Mixture of Multivariate Skew Laplace normal Distributions

Let us consider the definitions of [12], [11], [13] for Mixture of Multivariate Skew Laplace normal distributions. The probability density function of an m-component mixture model with parameter vector set (ξ, Ω, η); $\xi = \{\xi_1, \xi_2, \dots, \xi_m\}$ a set of vectors represent location parameters, $\Omega = \{\Omega_1, \Omega_2, \dots, \Omega_{\mathrm{m}}\}$ a set of covariance matrices, the shape parameter $\eta = \{\eta_1, \eta_2, \dots, \eta_{\mathrm{m}}\}$ is

$$\mathcal{P}(\mathrm{x}; \xi, \Omega, \eta, \varepsilon) = \sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\, \mathcal{P}(\mathrm{x}; \xi_i, \Omega_i, \eta_i) \tag{25}$$

where, $\varepsilon_{\mathrm{i}} \geq 0$, $\sum_{\mathrm{i}=1}^{\mathrm{n}} \varepsilon_{\mathrm{i}} = 1$, $\mathrm{f}(\mathrm{x}; \xi_i, \Omega_i, \eta_i)$ are defined as in (**8**)for a known $(\xi_i, \Omega_i, \eta_i)$, $i = 1, \dots, m$ and the notation $\mathrm{MMSLN_d}(\xi, \Omega, \eta, \varepsilon)$ represents mixture of multivariate skew Laplace distribution then for any j-th component density in (8) is obtained as

$$\mathrm{X_j}\left|\left(\kappa_{\mathrm{j}} = \mathrm{i}\right) \overset{\mathrm{d}}{=} \xi_{\mathrm{i}} + (\Omega_i)^{\frac{1}{2}}\left[\frac{\eta_i|\mathrm{U}_{2j}|}{\sqrt{\upsilon_i(\upsilon_i + \eta_i{}'\eta_i)}} + (\upsilon_i I_d + \eta_{\mathrm{i}}\eta_{\mathrm{i}}{}')^{\frac{1}{2}}\mathrm{U}_{1j}\right]\right. \quad , \ \mathrm{j} = 1,2, \dots, \mathrm{m} \tag{26}$$

where$\mathrm{U}_{1j} \sim \mathrm{MN_d}\left(\xi_j, \Omega_j\right)$, $\mathrm{U}_{2j} \sim \mathrm{N}(0,1)$ and for each $\mathrm{j} = 1,2, \dots, \mathrm{m}$, υ_j has the inverse gamma distribution with the probability function which defined in equation (**11**), Also, suppose that U_{1j}, U_{2j} and υ_j are mutually independent.

Equations ((**14**)-(**15**)) gives

$$(\mathbf{15})\mathrm{E(X)} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\left(\xi_i + \sqrt{\frac{2}{\pi}}(\Omega_i)^{\frac{1}{2}}\eta_i \Upsilon_i\right) \tag{27}$$

$$\mathrm{Cov(X)} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \varepsilon_{\mathrm{i}}\left((d+1)\Omega_i - \frac{2}{\pi}(\Omega_i)^{\frac{1}{2}}\ \eta_i\eta_i{}'(\Omega_i)^{\frac{1}{2}}\Upsilon_i{}^2 + \left(\xi_i + \sqrt{\frac{2}{\pi}}(\Omega_i)^{\frac{1}{2}}\eta_i\Upsilon_i\right)\left(\xi_i + \sqrt{\frac{2}{\pi}}(\Omega_i)^{\frac{1}{2}}\eta_i\Upsilon_i\right)'\right) - \mathrm{E(X)\,E(X)'} \tag{28}$$

Lemma 4. Let $\mathrm{X} \sim \mathrm{MMSLN_d}(\xi, \Omega, \eta, \varepsilon)$. Then

$$\mathrm{C_{lower}} \leq \mathrm{H_1}(\mathrm{X}; \xi, \Omega, \eta, \varepsilon) \leq \mathrm{C_{upper}} \tag{29}$$

where,

$$\mathrm{C_{upper}} = \frac{1}{2} \ln\left(\det(2\pi \exp(1)\mathcal{M})\right) \tag{30}$$

$$\mathrm{C_{lower}} = \sum_{\mathrm{i}=1}^{\mathrm{m}} -2\varepsilon_{\mathrm{i}} \ln\left(\frac{\det(\Omega_{\mathrm{i}})^{-\frac{1}{2}}}{2^{\mathrm{d}}\left(1+\eta_{\mathrm{i}}{}'\Omega_{\mathrm{i}}{}^{-1}\eta_{\mathrm{i}}\right)(\pi)^{\frac{1}{2}(\mathrm{d}-1)}\,\Gamma\left(\frac{\mathrm{d}+1}{2}\right)}\right) - \sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\left(2(\mathrm{d}+1)\eta_{\mathrm{i}}{}'\Omega_{\mathrm{i}}{}^{-1}\eta + \sqrt{\frac{2}{\pi}}\eta_{\mathrm{i}}{}'\Omega_{\mathrm{i}}{}^{-\frac{1}{2}}\eta_{\mathrm{i}}\Upsilon_i\right) - 2d \tag{31}$$

where,

$$\mathcal{M} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \varepsilon_{\mathrm{i}}\left((d+1)\Omega_i - \frac{2}{\pi}(\Omega_i)^{\frac{1}{2}}\ \eta_i\eta_i{}'(\Omega_i)^{\frac{1}{2}}\Upsilon_i{}^2 + \mathcal{K}_{\mathrm{i}}\mathcal{K}_{\mathrm{i}}'\right) \tag{32}$$

$$\mathcal{K}_{\mathrm{i}} = \xi_i + \sqrt{\frac{2}{\pi}}(\Omega_i)^{\frac{1}{2}}\eta_i\Upsilon_i - \mathrm{E(X)} \tag{33}$$

Proof: using lemma 1., we have

$$\mathrm{H}_1(\mathrm{X};\xi,\Omega,\eta,\varepsilon) \le \frac{1}{2}\ln\big(\det(2\pi\exp(1)\mathcal{M})\big)$$

From equation (**35**)(**25**)the Shannon entropy is

$$\mathrm{H}_1(\mathrm{X};\xi,\Omega,\eta,\varepsilon) = -\mathrm{E}\left(\ln\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\,\mathcal{P}(\mathrm{x};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})\right)\right)$$

Jensen's inequality gives

$$\mathrm{H}_1(\mathrm{X};\xi,\Omega,\eta,\varepsilon) \ge \sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\,\mathrm{H}_1(\mathrm{X};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})$$

Lemma 5. If $\mathrm{X}\sim\mathrm{MM\Omega LN}_{\mathrm{d}}(\xi,\Omega,\eta,\varepsilon)$, then

$$\mathrm{H}_{\boldsymbol{\tau}}(\mathrm{X};\xi,\Omega,\eta,\varepsilon) \le \mathfrak{C}_{\mathrm{Upper}} \tag{34}$$

where,

$$\mathfrak{C}_{\mathrm{Upper}} = \frac{1}{1-\boldsymbol{\tau}}\ln\left\{\exp\big((1-\boldsymbol{\tau})\mathrm{H}_{\boldsymbol{\tau}}(\mathrm{X};\xi_{\mathrm{m}},\Omega_{\mathrm{m}},\eta_{\mathrm{m}})\big) + \sum_{\mathrm{i}=1}^{\mathrm{m}-1}\left(\sum_{\mathrm{k}=1}^{\mathrm{i}}\varepsilon_{\mathrm{k}}\right)^{\boldsymbol{\tau}} \cdot \begin{pmatrix} \exp\big((1-\boldsymbol{\tau})\mathrm{H}_{\boldsymbol{\tau}}(\mathrm{X};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})\big) \\ -\exp\big((1-\boldsymbol{\tau})\mathrm{H}_{\boldsymbol{\tau}}(\mathrm{X};\xi_{\mathrm{i}+1},\Omega_{\mathrm{i}+1},\eta_{\mathrm{i}+1})\big)\end{pmatrix}\right\} \tag{35}$$

Proof: Mixture density in (**25**) implies that

$$\big(\mathcal{P}(\mathrm{x};\xi,\Omega,\eta,\varepsilon)\big)^{\boldsymbol{\tau}} = \left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\,\mathcal{P}(\mathrm{x};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})\right)^{\boldsymbol{\tau}}$$

Applying lemma 5. In [14] when $\mathrm{p}=\boldsymbol{\tau}$, we obtain

$$\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \varepsilon_{\mathrm{i}}\,\mathcal{P}(\mathrm{x};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})\right)^{\boldsymbol{\tau}} \ge \mathcal{P}(\mathrm{x};\xi_{\mathrm{m}},\Omega_{\mathrm{m}},\eta_{\mathrm{m}})^{\boldsymbol{\tau}} + \sum_{\mathrm{i}=1}^{\mathrm{m}-1}\left(\sum_{\mathrm{k}=1}^{\mathrm{i}}\varepsilon_{\mathrm{k}}\right)^{\boldsymbol{\tau}}\begin{pmatrix}\big(\mathcal{P}(\mathrm{x};\xi_{\mathrm{i}},\Omega_{\mathrm{i}},\eta_{\mathrm{i}})\big)^{\boldsymbol{\tau}} \\ -\big(\mathcal{P}(\mathrm{x};\xi_{\mathrm{i}+1},\Omega_{\mathrm{i}+1},\eta_{\mathrm{i}+1})\big)^{\boldsymbol{\tau}}\end{pmatrix}$$

Consequently,

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx \geq \int_{R^d}\mathcal{P}(x;\xi_m,\Omega_m,\eta_m)^{\boldsymbol{\tau}}dx + \sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^{\boldsymbol{\tau}}\int_{R^d}\begin{bmatrix}\left(\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{\boldsymbol{\tau}}\\ -\left(\mathcal{P}(x;\xi_{i+1},\Omega_{i+1},\eta_{i+1})\right)^{\boldsymbol{\tau}}\end{bmatrix}dx$$

Multiplying by $\frac{1}{1-\alpha}$ for both side, we have

$$\begin{aligned}&H_{\boldsymbol{\tau}}(X;\xi,\Omega,\eta,\varepsilon)\\ &\leq \frac{1}{1-\boldsymbol{\tau}}\ln\left\{\int_{R^d}\mathcal{P}(x;\xi_m,\Omega_m,\eta_m)^{\boldsymbol{\tau}}dx + \sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^{\boldsymbol{\tau}}\int_{R^d}\begin{bmatrix}\left(\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{\boldsymbol{\tau}}\\ -\left(\mathcal{P}(x;\xi_{i+1},\Omega_{i+1},\eta_{i+1})\right)^{\boldsymbol{\tau}}\end{bmatrix}dx\right\}\end{aligned}$$

Lemma 6. Let $X\sim MMSLN_d(\xi,\Omega,\eta,\varepsilon)$. Then, for each $k_1,k_2,\dots,k_m \in Z^+$ and $\sum_{i=1}^{m}k_i=\boldsymbol{\tau}$ the following approximation

$$\begin{aligned}&\frac{1}{\boldsymbol{\tau}}\ln\left\{\left(\frac{\boldsymbol{\tau}!}{k_1!\ k_2!\ \dots\ k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i}\right\}\\ &\quad\cong -\sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)}\right)\end{aligned}\tag{36}$$

is satisfied as $\boldsymbol{\tau}\to\infty$,such that $\gamma_j=\frac{k_j}{\boldsymbol{\tau}}, j=1,2,\dots,m$

Proof:

$$\begin{aligned}&\frac{1}{\boldsymbol{\tau}}\ln\left\{\left(\frac{\boldsymbol{\tau}!}{k_1!\ k_2!\ \dots\ k_m!}\right)\prod_{j=1}^{m}\left(\varepsilon_j\mathcal{P}(x;\xi_j,\Omega_j,\eta_j)\right)^{k_j}\right\}\\ &\quad= \frac{1}{\boldsymbol{\tau}}\ln(\boldsymbol{\tau}!) - \frac{1}{\boldsymbol{\tau}}\sum_{j=1}^{m}\ln(k_j!) + \frac{1}{\boldsymbol{\tau}}\sum_{j=1}^{m}k_j\ln\left(\varepsilon_j\mathcal{P}(x;\xi_j,\Omega_j,\eta_j)\right)\end{aligned}$$

Using the factorial approximation, we get

$$\begin{aligned}&\frac{1}{\boldsymbol{\tau}}\ln\left\{\left(\frac{\boldsymbol{\tau}!}{k_1!\ k_2!\ \dots\ k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i}\right\}\\ &\quad= \ln(\boldsymbol{\tau}) - 1 + \frac{1}{2\boldsymbol{\tau}}\ln(2\pi\boldsymbol{\tau}) - \frac{1}{\boldsymbol{\tau}}\sum_{i=1}^{m}k_I\ln(k_i) + \frac{1}{\boldsymbol{\tau}}\sum_{i=1}^{m}k_i - \frac{1}{2\boldsymbol{\tau}}\sum_{i=1}^{m}\ln(2\pi k_i)\\ &\quad+\sum_{i=1}^{m}\gamma_i\ln\left(\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)\end{aligned}$$

But $\gamma_i = \frac{k_i}{\boldsymbol{\tau}}$, $i = 1,2,\dots,m$, then $\sum_{i=1}^{m}\gamma_i = 1$. Consequently,

$$\frac{1}{\boldsymbol{\tau}}\ln\left\{\left(\frac{\boldsymbol{\tau}!}{k_1!\;\; k_2!\;\; \dots\;\; k_m!}\right)\prod_{i=1}^{m}\left(\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i}\right\} = \frac{1}{2\boldsymbol{\tau}}\left[\ln\left(\frac{(2\pi\boldsymbol{\tau})^{1-n}}{\prod_{i=1}^{n}\gamma_i}\right)\right] - \sum_{i=1}^{m}\gamma_i\ln\left(\frac{\gamma_i}{\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)}\right)$$

But $\lim_{\boldsymbol{\tau}\to\infty}\sum_{J=1}^{m}\gamma_J\ln\left(\frac{\gamma_J}{\varepsilon_J\mathcal{P}(x;\xi_J,\Omega_J,\eta_J)}\right) = 0$

Lemma 7. The approximation

$$H_{\boldsymbol{\tau}}(X;\xi,\Omega,\eta,\varepsilon) \cong \frac{1}{1-\boldsymbol{\tau}}\ln\left(\sum_{k_i\in B}\left(\prod_{i=1}^{m}(\gamma_i)^{-k_i}\right).\left(\prod_{i=1}^{m}\varepsilon_i^{k_i}\exp\left((1-\boldsymbol{\tau})H_{k_i}(X;\xi_i,\Omega_i,\eta_i)\right)\right)\right) \tag{37}$$

is satisfied as $\boldsymbol{\tau}\to\infty$.

where, $\sum_{k_J\in A}\frac{\boldsymbol{\tau}!}{\prod_{J=1}^{m}k_J!} = m^{\boldsymbol{\tau}}$, $A = \left\{k_J\in N, k_J > 0, \sum_{J=1}^{m}k_J = \boldsymbol{\tau}, J = 1,2,\dots,m\right\}$

Proof: from mixture model in (**25**), we have

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx = \int_{R^d}\left(\sum_{i=1}^{m}\varepsilon_i\,\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{\boldsymbol{\tau}}$$

Multinomial theorem gives us

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx = \int_{R^d}\sum_{k_i\in B}\frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}\left(\varepsilon_i\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i}dx \tag{38}$$

where, $\sum_{k_i\in A}\frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m}k_i!} = m^{\boldsymbol{\tau}}$, $A = \left\{k_J\in N, k_J > 0, \sum_{J=1}^{m}k_J = \boldsymbol{\tau}, j = 1,2,\dots,m\right\}$

replacing equation (**38**) in (**36**), we have

$$\int_{R^d} \left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}} dx \cong \int_{R^d} \sum_{k_i \in B} \exp\left\{-\boldsymbol{\tau} \sum_{i=1}^{m} \gamma_i \ln\left(\frac{\gamma_i}{\varepsilon_i \mathcal{P}(x;\xi_i,\Omega_i,\eta_i)}\right)\right\} dx$$

Then

$$\int_{R^d} \left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}} = \sum_{k_i \in B} \prod_{i=1}^{m} (\gamma_i)^{-k_i} \int_{R^d} \prod_{i=1}^{m} \left(\varepsilon_i \mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i} dx$$

Consequently,

$$\int_{R^d} \left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}} = \sum_{k_i \in B} \left[\prod_{i=1}^{m} (\gamma_i)^{-k_i}\right] \left[\prod_{i=1}^{m} \int_{R^d} \left(\varepsilon_i \mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i} dx\right]$$

Lemma 8. Consider $X \sim MMSLN_d(\xi,\Omega,\eta,\varepsilon)$, then

$$H_{\boldsymbol{\tau}}(X;\xi,\Omega,\eta,v,\varepsilon) \geq \mathfrak{C}_{Lower} \tag{39}$$

where,

$$\mathfrak{C}_{Lower} = \frac{1}{1-\boldsymbol{\tau}} \ln\left(\sum_{k_i \in B} \frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m} k_i!} \prod_{i=1}^{m} (\varepsilon_i)^{k_i} . \exp\left\{\frac{(1-\boldsymbol{\tau})}{\boldsymbol{\tau}} \sum_{i=1}^{m} k_i R_{\boldsymbol{\tau}}(X;\xi_i,\Omega_i,\eta_i)\right\}\right) \tag{40}$$

Proof: The Rényi entropy of $X \sim MMSLN_d(\xi,\Omega,\eta,\varepsilon)$ is

$$H_{\boldsymbol{\tau}}(X;\xi,\Omega,\eta,\varepsilon) = \frac{1}{1-\boldsymbol{\tau}} \ln\left(\int_{R^d} \left(\sum_{i=1}^{m} \varepsilon_i \, \mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{\boldsymbol{\tau}} dx\right)$$

By using multinomial theorem, we obtain

$$\int_{R^d} \left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}} dx = \sum_{k_i \in B} \frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m} k_i!} \prod_{i=1}^{m} (\varepsilon_i)^{k_i} \int_{R^d} \prod_{i=1}^{m} \left(\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{k_i} dx$$

Applying generalized HÖlder's Inequality, we have

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx \le \sum_{k_i\in B}\frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}(\varepsilon_i)^{k_i}\prod_{i=1}^{m}\left(\int_{R^d}\left(\mathcal{P}(x;\xi_i,\Omega_i,\eta_i)\right)^{p_ik_i}dx\right)^{\frac{1}{p_i}}$$

where, $p_1,p_2,\dots,p_m > 0$, $\sum_{i=1}^{m}\frac{1}{p_i} = 1$. Hence

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx \le \sum_{k_i\in B}\frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}(\varepsilon_i)^{k_i}\exp\left\{\sum_{i=1}^{m}\left(\frac{(1-p_ik_i)}{p_i}R_{p_ik_i}(X;\xi_i,\Omega_i,\eta_i)\right)\right\}$$

For each , $i = 1,2,\dots,m$, by choosing $p_i = \frac{\boldsymbol{\tau}}{k_i}$, where , $\sum_{i=1}^{m}\frac{1}{p_i} = \sum_{i=1}^{m}\frac{k_i}{\boldsymbol{\tau}} = 1$ and $1 \le \frac{\boldsymbol{\tau}}{k_i} \le \boldsymbol{\tau}$, we obtain

$$\int_{R^d}\left(\mathcal{P}(x;\xi,\Omega,\eta,\varepsilon)\right)^{\boldsymbol{\tau}}dx \le \sum_{k_i\in B}\frac{\boldsymbol{\tau}!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}(\varepsilon_i)^{k_i}\exp\left\{\frac{(1-\boldsymbol{\tau})}{\boldsymbol{\tau}}\sum_{i=1}^{m}k_iR_{\boldsymbol{\tau}}(X;\xi_i,\Omega_i,\eta_i)\right\}$$

Theorem 1. Let $X\sim \mathrm{MMSLN}_d(\xi,\Omega,\eta,\varepsilon)$. Then

$$\begin{aligned}
&H_\tau(X;\xi,\Omega,\eta,\varepsilon)\\
&= \frac{1}{2(1-\tau)}\left\{\ln\left(\sum_{k_i\in B}\frac{\tau!}{\prod_{i=1}^{m}k_i!}\prod_{i=1}^{m}(\varepsilon_i)^{k_i}.\exp\left\{\frac{(1-\tau)}{\tau}\sum_{i=1}^{m}k_iH_\tau(X;\xi_i,\Omega_i\right.\right.\right.\\
&+\ (1-\tau)H_\tau(X;\xi_m,\Omega_m,\eta_m)\\
&+\ln\left(\sum_{i=1}^{m-1}\left(\sum_{k=1}^{i}\varepsilon_k\right)^{\tau}.\left(\begin{array}{c}\exp\left((1-\tau)H_\tau(X;\xi_i,\Omega_i,\eta_i)\right)\\ -\exp\left((1-\tau)H_\tau(X;\xi_{i+1},\Omega_{i+1},\eta_{i+1})\right)\end{array}\right)\right)\Bigg\}
\end{aligned} \qquad (41)$$

Proof: from lemmas 11. and 14., we have the result.

The proof is directed from lemmas 11. and 14., by taking the mean of upper and lower bounds.

6. Conclusions and Final Remarks

We derive upper and lower bounds on the entropy in both types (Shannon and Rényi) of a multivariate skew Laplace normal random variable. Then, we extended these tools to the class of finite mixture of multivariate skew Laplace normal densities. Considering the average of these bounds, the approximate value of entropy can be calculated. Both entropies converge to a finite value of a multivariate skew Laplace normal random variable and its mixture model for any values of α order and dimension d. Given that mixture skews Laplace normal entropies is localized between the upper and lower bounds, the average of these bounds can be used as an approximation of the mixture skew Laplace normal entropies. In addition, the mixture skews Laplace normal entropy bounds provide useful information about the data and could be considered as a criterion to choose the possible number of components in each gender-based group.

Finally, we encourage researchers to use the proposed approach for real-world applications and data analysis, such as environmental [15], biological [4] data.

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