Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 3, No. 1, 2025, Pages 159-166
DOI: 10.47831/mjpas.v3i1.318

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage: https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

Applications of KKAT Transform Technique in Convolution Kind Linear Volterra Operator Equations

Ahmed Mahdi Abbood[1], Emad A. Kuffi[2*]

[1] Holy Karbala Education Directorate, Holy Karbala. Iraq

[2] Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

[*] Corresponding author E-mail: emad.kuffi@uomustansiriyah.edu.iq

Article Info.

Article history:

Received
15 September 2024

Accepted
23 September 2024

Publishing
30 January 2025

Abstract

Due to the importance of integral equations in engineering and scientific applications, there are many papers that have solved integral equations of both linear and nonlinear types.In this paper, Karry-Kalim Adnan transform (KKAT) is utilized to find the solution of convolution kind linear Volterra equations of the first and second kind. Examples are offered to illustrate the Karry-Kalim Adnan technique for solving convolution kind Volterra operator equations. It was noted that this technique is much easier in terms of mathematical operations and simplicity in arriving at the exact solution and this is explained through the applications in the paper.

The official journal published by the College of Education at Mustansiriya University

Keywords: Karry-Kalim Adnan transform; Volterra integral equations; Convolution kind Integral equations.

1. Introduction

Linear Volterra Integral equations of the 2^{nd} type is defined as:

$$V(\tau) = H(\tau) + \lambda \int_{z=0}^{\tau} k_e(\tau, z)\ V(z) dz$$

Where $V(\tau)$ is unknown function that will be determined, $k_e(\tau, z)$ is the kernel of the above equation and λ is any parameter [14, 16].The 1^{st} type linear Volterra integral equation is constructed by $H(\tau) = \int_0^{\tau} k_e(\tau, z)\ V(z) dz$.

The integral transform techniques are among the most widely utilized mathematical methods to evaluate the exact solutions of problems in physics, natural sciences and branches of engineering without large computation labor. There are many integral transformations to find the solution of integral equations such as. The Laplace transform is the most popular of these techniques. In [5] revealed how to apply the Sumudu transform to find the solution of integral equations of the convolution kind. In [18] evaluated convolution kind Volterra equations of 2^{nd} type and [13] obtained the exact solutions of convolution kind linear Volterra integral equations of first type via utilizing the Elzaki technique. In [17] proved how to solve the convolution kind linear Volterra equations via aid of Mohand technique. In [1-4] applied Kamal, Aboodh, Shehu and Mahgoub transforms techniques to find the solution of linear Volterra equations with an integral in the formula of a convolution. In [10] applied Tarig transformation technique to solve the convolution kind linear Volterra equations. In [11, 12] used Kharrat –Toma and Kashuri–Fundo transformations to find the solution of convolution kind linear Volterra equations. In [6-9] applied SEE and complex SEE transforms to solve the convolution of integral equations. An integral transform, the Karry–Kalim Adnan transform (KKAT) , was described by D.P.Patil , K.J.Patil and S.A. Patil [15] in this reference applications of ordinary differential equations, growth and decay problems were solved . They showed some properties and propositions of this transformation for ordinary differential equations. The purpose of this article is to apply the KKAT transform technique to find the exact solutions of convolution kind linear Volterra Computational work. The main goal of using this technique to solve all types of integral equations accurately is to reduce difficult calculations and make them simpler than direct methods.

There is no integral transform that is better than any other transform because each transform has the peculiarity of simplifying the mathematical operations according to the problem on the basis of which the transform was proposed. Every integral transform has an advantage in solving a specific application [21]. This application may have an exact solution that cannot be obtained except after difficult calculations. This is why many integral transforms with one parameter, two parameters, or three parameters have appeared, and here they are. These parameters play the essential role in arriving at an accurate and easy solution.

We will provide essential regarding the KKAT transform technique.

Definition of KKAT [15]

A transform defined for function of exponential order from set B.

B={g(x) : there exists p , $a_1, a_2 > 0$, $|g(x)| < p\ e^{\frac{|x|}{ai}}, x \in [0,\infty)\ , and\ i = 1,2\}$ (1)

Where p is a finite constant and a_1, a_2 may be finite or infinite.

KKAT is represented via operator $k\{.\}$ and is defined as:

$$k\{g(x)\} = \frac{1}{s}\int_0^\infty g(vx)\, e^{-sx} dx \quad , x \geq 0 \quad s, v\ \in [a_1, a_2]$$

Where s and v are parameters and v, s $\neq 0$

$k\{g(x)\}$. Can be written as:

$$k\{g(x)\} = \frac{1}{sv}\int_0^\infty g(x)\, e^{-\frac{s}{v}x} dx = F\left(\frac{s}{v}\right) \tag{2}$$

Also

$$g(x) = k^{-1}\left\{F\left(\frac{s}{v}\right)\right\} = \frac{1}{2\pi j}\int_{\epsilon - j\infty}^{\epsilon + j\infty} svF(\frac{s}{v})\, e^{-\frac{s}{v}x} ds$$

$$j \in \mathbb{C} \quad where\ \mathbb{C}\ is\ the\ complex\ numbers\ set\ .$$

Some important formulae of KKAT integral transform, [15]

Function $g(t)$	$k\{g(x)\} = \frac{1}{sv}\int_0^\infty g(x)\, e^{-\frac{s}{v}x} dx = F\left(\frac{s}{v}\right)$.
c	$\frac{c}{s^2}$, c is constant
x^n , $n \epsilon N$	$\frac{n!\, v^n}{s^{n+2}}$
e^{cx}	$\frac{1}{s(s - cv)}$, c is constant
1^{st} derivative $\acute{g}(x)$	$k\{\acute{g}(x)\} = \left(\frac{s}{v}\right) k\{g(x)\}n - \frac{g(0)}{sv}$.
2^{nd} derivative	$k\{g''(x)\} = (\frac{s}{v})^2 k\{g(x)\} - \frac{\acute{g}(0)}{sv} - \frac{g(0)}{v^2}$

$g''(x)$	
$e^{cx}g(x)$	$\frac{(s-cv)}{v}F.(v,s-cv).$
$\cos(x)$	$\frac{1}{s^2+v^2}$
$\sin(x)$	$\frac{v}{s(s^2+v^2)}$
$\cosh(x)$	$.\frac{1}{s^2-v^2}.$
$\sinh(x)$	$\frac{v}{s(s^2-v^2)}$

Application of KKAT on the integral function [15]

If $g(x)=\int_0^x f(z)\,dz$ $, then\ k\left\{\int_0^x f(z)\,dz\right\}=\left(\frac{v}{s}\right)F\left(\frac{s}{v}\right).$

Now in this theorem we will prove the convolution to KKAT transform:

Theorem (1.1): (Convolution Theorem)

Let $g_1(x)\ and\ g_2(x)$ have integral transform $F_1\left(\frac{s}{v}\right)\ and\ F_2\left(\frac{s}{v}\right).$

then the KKAT integral transform of the convolution of g_1 and g_2 is

$$g_1*g_2=\int_0^x g_1(x)\,.\,g_2(x-\tau)d\tau=svF_1\left(\frac{S}{v}\right).F_2\left(\frac{S}{v}\right).$$

Proof:

$$K\{g_1*g_2\}=\frac{1}{sv}\int_0^\infty e^{-\frac{s}{v}x}\int_0^\infty g(x)\,.\,g_2(x-\tau)dx\ ,$$

$$=\frac{1}{sv}\int_0^\infty g_1(\tau)\,d\tau\int_0^\infty e^{-\frac{s}{v}x}\,.\,g_2(x-\tau)dx\ ,$$

$$=\frac{1}{sv}\int_0^\infty e^{-\frac{s}{v}x}\,g_1(\tau)\,d\tau\int_0^\infty e^{-\frac{s}{v}x}v\,g_2(x)\,dx=svF_1\left(\frac{S}{v}\right).F_2\left(\frac{S}{v}\right).$$

2. Main Results

In this paper, we focus on the convolution kind kernel $k_e(\tau.z)$ which is expressed as the difference $(\tau-z)$ .The convolution kind LVIESK ($2^{nd}\ kind$)has the formula:

$V(\tau)=H(\tau)+\lambda\int_0^\tau k_e\,(\tau-z)V(z)\,dz.$

And the convolution kind LVIEFK ($1^{st}\ kind$) is written as:

$H(\tau)=\int_0^\tau K_e(\tau-z)\ V(z)dz.$

Theorem (2.1): The solution of convolution kind LVVIEFK

$$H(\tau)=\int_0^\tau K_e(\tau-z)V(z)\,dz \qquad (3)$$

is given as:

$$V(\tau) = K^{-1}\left\{F\left(\frac{S}{v}\right)\right\} = K^{-1}\left\{\frac{1}{vs}\frac{K\{H(\tau)\}}{K\{k_e(\tau)\}}\right\}.$$

Where K_e is the kernel and $\mathrm{K}\{\mathrm{V}(\tau)\} = F\left(\frac{s}{v}\right)$.

Proof:

We can write

$$k\{H(\tau)\} = k\left\{\int_0^{\tau} k_e\,(\tau - z)\,V(z)\,dz\right\},$$

$$k\{H(\tau)\} = k\{k_e(\tau)\ * V(\tau)\}.$$

By taking KKAT to either side of eq. (3). By implementing convolution theorem for KKAT, we obtain:

$$k\{H(\tau)\} = SV\ k\{k_e(\tau)\}.k\{V(\tau)\},$$

$$k\{V(\tau)\} = \frac{1}{sv}.\frac{k\{H(\tau)\}}{k\{k_e(\tau)\}} \tag{4}$$

Having used the inverse KKAT on either side of eq.(4) , we get

$$V(\tau) = k^{-1}\left\{F\left(\frac{S}{v}\right)\right\} = k^{-1}\left\{\frac{1}{vs}\frac{k\{H(\tau)\}}{k\{k_e(\tau)\}}\right\}\ .$$

This represents the desired solution.

Theorem (2.2)

The solution of convolution kind LVIESK

$$V(\tau) = H(\tau) + \lambda\int_0^{\tau} k_e(\tau - z)\,V(z)\,dz \tag{5}$$

Is given as:

$$V(\tau) = k^{-1}\left\{F\left(\frac{S}{v}\right)\right\} = k^{-1}\left\{\frac{k\{H(\tau)\}}{1 - \lambda sv\ k\{k_e(\tau)\}}\right\}.$$

Where $k_e$is the kernel and $k\{V(\tau)\} = F\left(\frac{s}{v}\right)$.

Proof:

Taking the KKAT on either side of VIESK eq.(5) , we get:

$$k\{V(\tau)\} = k\left\{H(\tau) + \lambda\int_0^{\tau} k_e\,(\tau - z)\,V(z)\,dz\right\},$$

$$k\{V(\tau)\} = k\{H(\tau)\} + \lambda k\left\{\int_0^{\tau} k_e\,(\tau - z)\,V(z)\,dz\right\},$$

$$k\{V(\tau)\} = k\{H(\tau)\} + \lambda k\left\{\int_0^{\tau} k_e(\tau) * V(\tau)\right\}.$$

We find the following expressions:

$$k\{V(\tau)\} = k\{H(\tau)\} + \lambda sv\ k\{k_e(\tau)\}\,.\,k\{V(\tau)\}$$

$$k\{V(\tau)\} = \frac{k\{H(\tau)\}}{1 - \lambda sv\ k\{k_e(\tau)\}} \tag{6}$$

By using convolution theorem for KKAT. Having used the inverse KKAT on either side of eq.(6) , we get the solution is

$$V(\tau) = k^{-1}\left\{\frac{k\{H(\tau)\}}{1-\lambda sv\ k\{(k_e(\tau))\}}\right\}.$$

3. Application

This part explains how to find the exact solution of convolution kind linear Volterra equations utilizing the KKAT by some important applications. These applications were taken from the reference [20] and applied to KKAT technique.

Application (3.1)

Apply the KKAT technique to find the solution convolution kind LVIEFK:

$$\sinh(x) = \int_0^x e^{x-z}\ V(z)\,dz.$$

Let's taken $\mathrm{k}\{\mathrm{V(x)}\} = \mathrm{F}\left(\frac{\mathrm{s}}{\mathrm{v}}\right)$. It is obtained that :

$$k\{\sinh(x)\} = k\{\int_0^x e^{x-z}\ .V(z)\,dz\},$$

$$\frac{v}{s(s^2-v^2)} = k\{e^x * V(x)\} \tag{7}$$

By implementing the KKAT. Utilizing convolution theorem for KKAT on eq.(7) , we get:

$$\frac{v}{s(s^2-v^2)} = sv\ k\{e^x\}.k\{V(x)\},$$

$$\frac{v}{s(s^2-v^2)} = sv\ .\frac{1}{s(s-v)}\ .F\left(\frac{s}{v}\right),$$

$$F\left(\frac{s}{v}\right) = \frac{1}{s(s+v)}.$$

Therefore, we write

$$k\{V(x)\} = F\left(\frac{s}{v}\right) = \frac{1}{s(s-v)} \tag{8}$$

Having using the inverse KKAT on either side of eq. (8), we obtain:

$$V(x) = k^{-1}\left\{\frac{1}{s(s+v)}\right\} = e^{-x}\ .$$

Consequently, we arrive at the answer as:

$$V(x) = e^{-x}.$$

Application (3.2)

Evaluate the solution of convolution kind LVIESK

$$V(x) = \cos(x) - \int_0^x (x-z)\cos(x-z)\,V(z)\,dxz.$$

Applying the KKAT technique.

Assume that $k\{V(x)\} = F\left(\frac{s}{v}\right)$. Having used the KKAT

$$k\{V(x)\} = k\{\cos(x)\} - k\{\int_0^x (x-z)\cos(x-z)\,V(z)\,dxz\},$$

$$= k\{\cos(x)\} - k\{x\cos(x) * V(x)\}\,.$$

Now, by implementing convolution property for KKAT, it is found as:

$$F\left(\frac{S}{v}\right) = k\{\cos(x)\} - sv\; k\{x\cos(x)\}\, k\{V(x)\}\,,$$

$$F\left(\frac{S}{v}\right) = \frac{1}{s^2+v^2} - sv\;\left(-v\;\frac{\partial k}{\partial s}\{\cos(x)\} + sv\; k\{\cos(x)\}\right) F\left(\frac{S}{v}\right),$$

$$F\left(\frac{S}{v}\right) = \frac{1}{s^2+v^2} + \frac{s^4v^4 - s^2v^2}{(1+s^2v^2)^2} F\left(\frac{S}{v}\right).$$

Therefore, we get:

$$k\{V(x)\} = F\left(\frac{S}{v}\right) = \frac{1+s^2v^2}{1+3v^2s^2}\quad .$$

Take the inverse of KKAT, we get the exact solution is

$$V(x) = \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}x)\quad .$$

Application (3.3)

Take the I.V.P (initial value problem)

$$\begin{cases} \acute{\acute{V}}(x) - 2\acute{V}(x) - 3V(x) = 0 \\ V(0) = 1 \qquad , \quad \acute{V}(0) = 2 \end{cases}$$

This is equivalently to Volterra equation:

$$V(x) = 1 + \int_0^x (3x - 3z + 2)\; V(z)\, dz.$$

The above equation can be expressed as

$$V(x) = 1 + 3\int_0^x (x-z)\,V(z)\,dz + 2\int_0^x V(z)\,dz \tag{9}$$

If we use KKAT on either side of eq.(9), it is found as:

$$k\{V(x)\} = k\{1\} + 3k\{x * V(x)\} + 2k\{1 * V(x)\}\;\; .$$

Suppose that $k\{V(x)\} = F\left(\frac{s}{v}\right)$. Considering the convolution property for KKAT, we get:

$$F\left(\frac{S}{v}\right) = \frac{1}{s^2} + 3sv\; k\{x\}.\, k\{V(x)\} + sv\; k\{1\}.\, k\{V(x)\}.$$

$$F\left(\frac{S}{v}\right) = \frac{1}{s^2} + 3\frac{v^2}{s^2}\; F\left(\frac{S}{v}\right) + \frac{2v}{s} F\left(\frac{S}{v}\right).$$

Therefore, we obtain:

$$k\{V(x)\} = F\left(\frac{s}{v}\right) = \frac{\frac{1}{s^2}}{\left(1 - \frac{3v^2}{s^2} - \frac{2v}{s}\right)} = \frac{1}{s^2\left(\frac{s^2 - 3v^2 - 2sv}{s^2}\right)}.$$

Then having used the inverse KKAT, and after simple computation, we get:

$$k^{-1}\left\{F\left(\frac{s}{v}\right)\right\} = \frac{1}{4}k^{-1}\left\{\frac{1}{s(s+v)}\right\} + \frac{3}{4}k^{-1}\left\{\frac{1}{s(s-3v)}\right\}.$$

Consequently, we have the solution as:

$$V(x) = \frac{1}{4}(e^{-x} + 3e^{3x}).$$

4. Conclusion

The KKAT technique is applied to evaluate the exact solution of the Volterra integral equations (VIE's) of the convolution kind. The presented applications illustrate that the KKAT gives the exact solution of the convolution kind equations as the other integral transformation, and further requiring less time and effort in computational work to get exact solution of operator equations. KKAT technique: through the problems above, we noticed the ease of solving all types of integral equations.

Reference

[1] S. Aggarwal, R. Chauhan, and N. Sharma, “Application of Aboodh transform for solving linear Volterra Integro-differential equations of second kind,” *Int. J. Res. Adv. Tech.*, vol. 6, no. 8, pp. 1186-1190, 2018.

[2] S. Aggarwal, R. Chauhan, and N. Sharma, “A New Application of Mahgoub Transform for solving linear Volterra Integral Equations,” *Asia. Res.*, vol. 7, no. 2, pp. 46-48, 2018.

[3] S. Aggarwal, R. Chauhan, and N. Sharma, “A New Application of Kamal Transform Solving Linear Integral Equations,” *Int. J. Lat. Tech. Eng., Magn. And App Sci.*, vol. 7, no. 4, pp. 138-140, 2018.

[4] S. Aggarwal, R. Chauhan, and N. Sharma, “A New Application of Shehu Transform for Handling Volterra Integral Equations of first kind,” *Int. J. Res. Advent Technol.*, vol. 7, no. 4, pp. 439-445, 2019.

[5] M. A. Asiru, “Sumudu Transform and the solution of Integral Equations of convolution Type,” *Int. J. Math. Educ. Sci. Technol.*, vol. 32, no. 6, pp. 906-910, 2001.

[6] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, “Complex SEE Integral transform in solving Abel’s integral equation,” *J. Interdiscip. Math.*, vol. 25, no. 5, pp. 1307-1314, 2022.

[7] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, “The solution of Faltung Type Volterra Integro-Differential Equation of First kind Using Complex SEE Transform,” *J. Coll. Educ. Mustansiriyah Univ.*, vol. 23, no. 1, 2022.

[8] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, “Applying SEE transform to solve linear Volterra integro-differential equation of the second kind,” *AIP Conf. Proc.*, vol. 2591, no. 1, 2023.

[9] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, “Solving Abel’s Integral Equation Using (Sadiq–Emad–Eman) Transform,” *IEEE Math.*, pp. 251-254, 2021.

[10] M. G. Gnanavel, C. Saranya, and A. Viswanathan, “Application of linear Volterra Integral Equations of First kind by Using Tarig Transform,” *Int. J. Innov. Technol. Explor. Eng.*, vol. 8, no. 10, pp. 2278-3075, 2019.

[11] N. Gungor, “Solving convolution Type linear Volterra Integral Equations with Kashuri Fundo Transform,” *J. Abstr. Comput. Math.*, vol. 6, no. 2, pp. 1-7, 2021.

[12] N. Gungor, “Application of Kharrat–Toma Transform for solving Linear Volterra Integral Equations,” *J. Univ. Math.*, vol. 3, no. 2, pp. 137-145, 2021.

[13] P. Haara, “On Volterra integral equations of the first kind by using Elzaki transform,” *Far East J. Math. Sci.*, vol. 102, no. 9, pp. 1857-1863, 2017.

[14] M. Rahman, *Integral Equations and Their Applications*, Boston: WIT Press, 2007.

[15] D. P. Patil, K. J. Patil, and S. A. Patil, “Applications of Karry-Kalim Adnan Transformation (KKAT) in Growth and Decay problems,” *Int. J. Innov. Res. Technol.*, vol. 9, no. 7, pp. 437-442, 2022.

[16] A. M. Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Berlin Heidelberg: Springer Verlag, 2011.

[17] P. Senthil Kamar, M. G. Gnanavel, and V. Viswanathan, “Application of Mohand Transform for solving linear Volterra Integro-Differential Equations,” *Int. J. Res. Adv. Technol.*, vol. 6, no. 10, pp. 2554-2556, 2018.

[18] Y. Song and G. Kim, “The solution of Volterra Integral Equation of the second kind by Using the Elzaki Transform,” *Appl. Math. Sci.*, vol. 8, no. 11, pp. 525-530, 2014.

[19] K. Iqbal, M. Kalim, and A. Khan, “Applications of Karry-Kalim Adnan Transformation (KKAT) to Mechanics and Electrical Circuits,” *Hindawi J. Funct. Space*, 2022.

[20] N. Gungor, “Solution of Convolution Type Linear Volterra Integral Equations with Formable Transform,” *Int. J. Latest Technol. Eng. Appl. Sci.*, 2022.

[21] H. Jafari, “A new general integral transform for solving integral equations,” *J. Adv. Res.*, vol. 32, pp. 133-138, 2021.