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y-Closed-Pseudo-Projective Modules

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A B S T R A C T

In this work, we present new concept which is y-closed-pseudo-projective module (briefly, YCP-projective module). This work which is generalization of pseudo-projective modules. We have provided some characteristics and descriptions of those concepts. Semi-simple modules have been characterized in terms of YCP- projective modules. We have shown the relationships of YCP-projective with other concepts, including a Co-Hopfian, directly finite modules.

Keywords: Y-closed-pseudo-projective, Y-closed submodule, YCP-K-projective module, YCPP-module, Co-Hopfian.

1. Introduction

Throughout this work, R is a ring with identity, and each R-module is a unitary right R-module, $A \subseteq P$ denotes A is a submodule of an R-module P, $Hom_R(P,K)$ ($Epi_R(P,K)$) denotes all an R-homomorphism (R-epimorphism) from P to R- module K over ring R.
Let P and K be R-modules. P is referred to as (pseudo)-K-projective if for any $\gamma \in Hom_R(P, K/B)$ ($Epi_R(P, K/B)$) where $B \subseteq K$ there exists $\delta \in Hom_R(P,K)$ with $\pi \delta = \gamma$, where $\pi: K \longrightarrow K/B$ be the natural R-epimorphism. An R-module K is a quasi-projective, if K is a K-projective. Also, P is a projective if it is K-projective for all R-module K. (see [1– 4]).
A submodule B of an R-module K is said to be closed in K (briefly, $B \subseteq_c K$) if B has no proper essential extension inside K. The submodule Z (B) of K define as $Z(B) = \{b \in B: ann(b) \subseteq_e R\}$ is called singular of K. If $Z(K) = K$ ($Z(K) = 0$), then K is a singular (nonsingular). For a submodule B is said to be y-closed (briefly, $B \subseteq_{yc} K$) if K/B be a

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A.A. Ahmed and Mahdi Saleh Nayef in [10], presented the concept of pseudo -y- closed -injective modules. Also, B. H. Al-Bahrani in [7], introduces the concept of y-closed-projectivity. Let P, K be R- modules. An R-module P is referred to as K-y-closed-projective (briefly, P is K-yc-projective), if for every $\beta \in Hom_R(P, K/B)$, where $B \subseteq_{yc} K$, $\exists\ \alpha \in$

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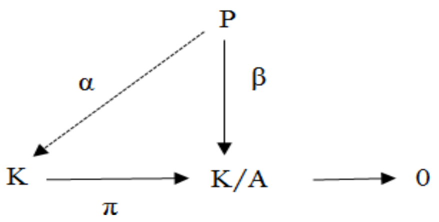
Hom $_R$(P, K) with π α = β with π be the natural R-epimorphism. An R-module P is yc-projective if P is a K-yc-projective, for any R-module K.

An R-module K is said to be directly finite if it is not isomorphic to a proper direct summand of K. For an R-module K is a co-Hopfian (Hopfian) if any monomorphism (epimorphism) in End $_R$(K) is an automorphism, see [8]. An R-module P is said to have D_2, if for each submodule B of P where P ⁄ B ≅ X with X ⊆⊕ P, then B ⊆⊕ P, see [1].

2. y-closed-Pseudo-Projective Modules.

We will present the concept of an YCP-K-projective module. This concept is a generalization of a pseudo-projective module.

Definition (2.1): Let K be an R-module. An R-module P is called y-closed-pseudo-K-projective (briefly P is YCP-K-projective) if for any y-closed submodule A of K and any β∈ Epi R(P, K ⁄ A), there exists α ∈ Hom R(P, K) such that π α = β. Where π be the natural R-epimorphism, i.e., the following diagram:



is commute.

Also, an R- module H is called YCPP-module, if H is an YCP-H-projective. Two modules H and D are said to be mutually YCP-projective if H is an YCP-D-projective, D is an YCP-H-projective

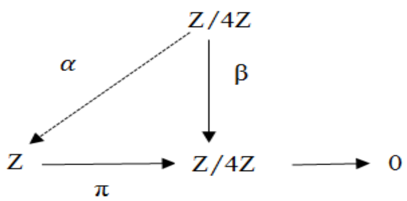
Examples and Remarks (2.2):

1) Every singular R-module is YCP-K-projective, for any R-module K.

Proof: Let P be a singular R-module. Let A $\subseteq_{yc}$ K and β ∈ Epi $_R$(P, K/A). Since K/A is a nonsingular, by [5, Proposition (1.20), p.31] we have β = 0. Therefore, there exists 0 = α ∈ Hom $_R$(P, K) with π α = β, where π is the natural R-epimorphism

2) Every pseudo-K-projective module is an YCP-K-projective. The opposite is not true.

Proof: A Z-module P = Z/4Z be an YCP-K-projective, since P is singular. Now it require to show that P is not pseudo-Z-projective. Suppose that P is pseudo-Z-projective. Let β: P⟶P, be defined by β (a + 4Z) = a + 4Z, where a ∈ Z, easily seen that β is a Z-epimorphism. Consider the illustration below**:**



There exist α ∈ Hom Z (P, Z) such that π α (n) = β (n), n ∈ P. But Hom Z (P, Z) = 0 by [5, Proposition (1.20), p.31]. It is follows that for all a ∈ Z, β (a + 4Z) = 4Z, which is a contradiction. So, P is not pseudo-Z-projective.

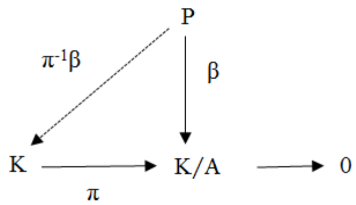
3) Clearly, any K-yc-projective module which is an YCP-K-projective, we know that any K-projective is K-yc-projective. Hence, any K-projective module is an YCP-K-projective.

4) Any simple R-module is YCP-K-projective.

Proof: Let P be a simple R-module, K be any R-module. By [5, Proposition (1.24)], P is singular or projective. Now, if P is singular, thus by (1) it is YCP-K-projective. If P is projective, then by (3) it is YCP-K-projective as well.

5) For an R-module K is simple y-closed R- module, if (0) and K are only y-closed submodule of K. Consider the module Z_2 as Z-module, clear that it is simple, but Z_2 is singular, thus by[9] we get Z_2 is only y-closed submodule of Z_2. Hence, Z_2 is not simple y-closed Z-module. We know that Z as Z-module is not simple, Z and (0) be only y-closed submodules of Z see [9], therefore, Z is simple y-closed Z-module. This means there are no relationship between simple R-module and simple y-closed R-module.
6) For simple y-closed R- module K, each R- module P is an YCP-K-projective.

Proof: Assume that P be an R-module. Let $\beta \in \text{Epi}_R(P, K/A)$ with $A \subseteq_{yc} K$. Consider the illustration below**:**



Since K is a simple R-module we have A=0 or A=K. If A=0, since ker (π) =A, therefore, π is an R-isomorphism. So, π^{-1} exists. It is follow that $\pi^{-1}\beta \in \text{Hom}_R(P, K)$ such that $\pi\pi^{-1}\beta = \beta$. Now, if A=K, clearly, P is an YCP-K-projective.

Definition (2.3): An R-module K is referred to as fully y-closed (briefly, K is FYC-module), if each submodule of K is y-closed of K.

Example (2.4): If D is a nonsingular semi-simple R-module, then D is a fully y-closed.

Proof: Let $A \subseteq D$, we have $A \subseteq_{\oplus} D$ since D is semi-simple, we know that every direct summand is closed, thus $A \subseteq_c D$. But D is nonsingular, hence $A \subseteq_{yc} D$ by [5, Proposition (2.4)].

Example (2.5): D = Z_6 as Z_6-module is fully y-closed, because it is evident that D is semi-simple and nonsingular. Also, Z as Z-module is not fully y-closed.

In the following result, we demonstrate that for a fully y-closed module, the concepts of the pseudo-K-projective and YCP-K-projective are equivalents.

Proposition (2.6): Let K be an R-module. If K is a FYC-module, then the following statements are equivalent:

1) pseudo-K-projective module;
2) YCP-K-projective module.

Proof: (1) ⟹ (2). Obviously.

(2) ⟹ (1). Assume that P is an YCP-K-projective. Let $A \subseteq K$ and let $\beta \in \text{Epi}_R(P, K/A)$. Since K is a fully y-closed, we have $A \subseteq_{yc} K$. By YCP-K-projectivity of P, there exists $\alpha \in \text{Hom}_R(P, K)$ with $\pi\alpha = \beta$. Hence, P is a pseudo-K-projective.

Proposition (2.7): Let P and K be two R- modules. If $\ker(\beta) \subseteq_{yc} P$ with any $\beta \in \text{Epi}_R(P, K/A)$, where A be any submodule of K, the next statements are equivalent:

1) P is a pseudo-K-projective;
2) P is an YCP-K-projective.

Proof: (1) ⟹ (2). Clear

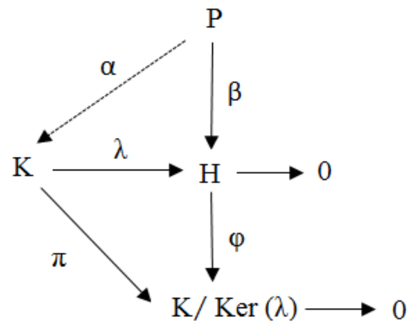
(2) ⟹ (1). Assume that P is an YCP-K-projective. Let β ∈Epi $_R$(P, K⁄A) and let π: K⟶ K⁄A be the natural R-epimorphism. By the first isomorphism theorem, we have K⁄A ≅ P⁄ ker (β). Since ker (β) ⊆yc P, Therefore, P⁄ ker (β) is a nonsingular. Therefore, K ⁄A is a nonsingular, hence A ⊆yc K. thus by (1), there exists α∈ Hom $_R$(P, K) with π α = β.

In the following proposition, we provide a characterization of an YCP-K-projective module.

Proposition (2.8): For R-modules P and K, the next statements are equivalent:

1) P is an y-closed-pseudo-K-projective;
2) For any λ ∈ Epi $_R$(K, H) with ker (λ) ⊆yc K, where H be each R-module , each β∈ Epi $_R$(P, H) there exists α ∈ Hom $_R$(P, K) with λ α = β.

Proof: (1) ⟹ (2). Let λ ∈ Epi $_R$(K, H) with ker (λ) ⊆yc K, let β∈ Epi $_R$(P, H). By the first isomorphism theorem, we have H ≅ K⁄ ker (λ), therefore, there exists an R-isomorphism φ: H⟶ K/ ker (λ) defined by φ (h) = m + ker (λ) where λ (m) = h. Consider the illustration below:

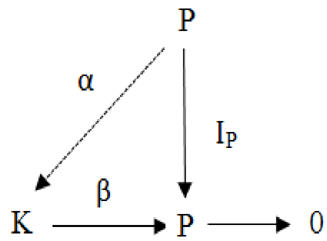


Clearly, φ β is an R-epimorphism. By (1), there exist α ∈ Hom $_R$(P, K) such that π α = φ β, where π: K⟶K/ker (λ) is the neutral R-epimorphism. For any m ∈ K, we have φ λ (m) = φ (λ (m)) = φ (h) =m+ ker (λ) = π (m). So, φ λ =π. Therefore, φ β = π α = φ λ α. Hence, λ α = β.

(2) ⟹ (1). It is clear.

Proposition (2.9): Let P is an YCP-K- projective and β: K ⟶ P be any R-epimorphism with ker (β) ⊆yc K, then ker (β) ⊆⊕ K.

Proof: Let I_P is the identity map of P. Consider the illustration below:



By Proposition 3.8, there exists α ∈ Hom $_R$(P, K) such that β α = I_P. Therefore β split. Hence, ker (β) ⊆⊕ K

In [2, (5.3.2)], an R-module P is called projective if any R-epimorphism β: K⟶P split, for all R-module K. In [4, Theorem 3.2], it was proved that; P is projective iff P is pseudo-K projective, for all R-module K. The next result is generalization of Theorem 3.2 in [4]

Proposition (2.10): Let R be a ring such that any R-module is fully y-closed, the next are equivalent:

1) YCP-K-projective module, for all R-module K.
2) projective module
3) yc-projective module

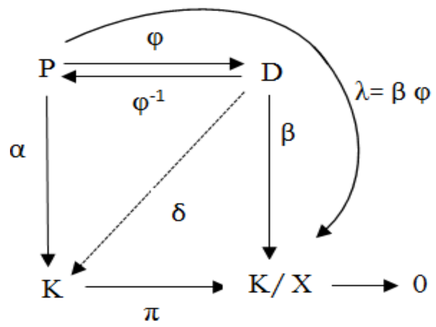
Proof: (1) $\Longrightarrow$ (2). Assume that P is pseudo -K- yc -projective such that K be any R-module. Let β: K$\longrightarrow$P be any R-epimorphism. Since K is fully y-closed R-module, so ker (β) $\subseteq_{yc}$ K. Therefore by (prop 2.8) we have β split.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). Clear.
The next result, we give a condition under which an YCP-K-projective is CLS-module.

Proposition (2.11): Let K be an R-module. If K/A is an YCP-K-projective module, for all y-closed submodules A of K, then K is CLS- module.

Proof: Assume that A $\subseteq_{yc}$ K. Let π: K$\longrightarrow$ K/A be the natural R-epimorphism. Hence, ker (π) = A. But K/A is an YCP-K-projective. Therefore, by Proposition (2.9) we get A is a direct summand of K. Hence, K is a CLS-module.
Now, we give some properties of YCP-K-projective module.

Proposition (2.12): If D $\cong$ P and P is an YCP-K-projective, then D is an YCP-K-projective.

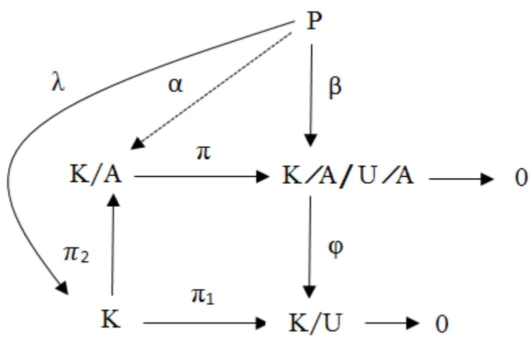
Proof: Let P is an YCP-K-projective and D $\cong$ P. Let X $\subseteq_{yc}$ K and $\beta \in$Epi $_R$(D, K/ X). Since D $\cong$ P, there exists an R-isomorphism φ: P$\longrightarrow$ D. Consider the illustration below:



It is clear that $\lambda = \beta\ \varphi \in$ Epi $_R$(P, K/X). Since P is an YCP-K-projective, there exists $\alpha \in$ Hom $_R$(P, K) with $\pi\ \alpha = \lambda$. Now, let $\delta = \alpha\ \varphi^{-1}$, we have $\pi\ \delta = \pi\ \alpha\ \varphi^{-1} = \beta\ \varphi\ \varphi^{-1} = \beta$. Hence, D is an YCP-K-projective.

Proposition (2.13): Let P be an YCP-K-projective. If A is a submodule of K, then P is an YCP-K/A-projective.

Proof: Let U/A is an y-closed submodule of K/A and $\beta\in$ Epi $_R$(P, K/A/ U/A). But K/A/ U/A $\cong$ K/U by third isomorphism theorem. Therefore, there exists an R-isomorphism φ: K/A/U /A $\longrightarrow$ K/U defined by φ **(**k +A + U/A**)** = k + U, for all k $\in$ K. Consider the illustration below:

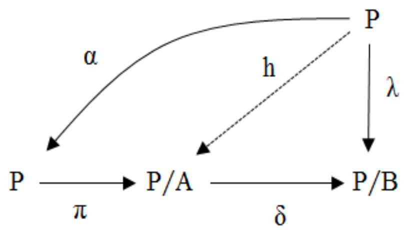


Where π , π_1, π_2 are the natural R-epimorphisms. Since K/A/ U/A is nonsingular, K/U is also non-singular, it follows that U $\subseteq_{yc}$ K. Since P is an YCP-K-projective, there exists $\lambda \in$ Hom $_R$(P, K) such that $\pi_1\ \lambda = \varphi\ \beta$. Let $\alpha = \pi_2\ \lambda$. Then $\varphi\ \beta = \pi_1\ \lambda = \varphi\ \pi\ \pi_2\ \lambda = \varphi\ \pi\ \alpha$, so $\beta = \pi\ \alpha$ since φ is an R-isomorphism. Therefore, P is an YCP-K/A-projective.
The next result gives a characterization of YCPP-module.

Proposition (2.14): Let P be a fully y-closed R-module, the next are equivalent:

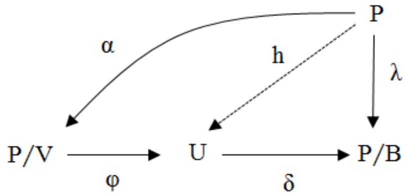
1) P is an YCPP-module;
2) For submodules A, B of P and R-epimorphisms $\delta: P/A \longrightarrow P/B$ and $\lambda: P \longrightarrow P/B$ there exists $h \in Hom_R(P, P/A)$ such that $\delta h = \lambda$;
3) For any submodule B and direct summand U of P with $\delta \in Epi_R(U, P/B)$ and $\lambda \in Epi_R(P, P/B)$ there exists $h \in Hom_R(P, U)$ with $\delta h = \lambda$.

Proof: (1) $\Longrightarrow$ (2). Let A, B be a submodules of P and $\delta \in Epi_R(P/A, P/B)$, $\lambda \in Epi_R(P, P/B)$. Consider the illustration below:



Clearly, $\delta \pi$ is an R-epimorphism and ker $(\delta \pi) \subseteq_{yc} P$ since P is a fully y-closed. Therefore, by Coro. (2.9), there exists $\alpha \in End_R(P)$ such that $\delta \pi \alpha = \lambda$. Now, let $h = \pi \alpha$, then $h \in Hom_R(P, P/A)$ with $\delta h = \delta \pi \alpha = \lambda$

(2)$\Longrightarrow$ (3). Let $U \subseteq_{\oplus} P$ and B be a submodule of P with $\delta \in Epi_R(U, P/B)$, $\lambda \in Epi_R(P, P/B)$. Consider the illustration below:



Since $U \subseteq_{\oplus} P$, there exists a submodule V of P such that $U \oplus V = P$. It follows that $P/V = U + V/V \cong U$ by second isomorphism theorem. So, there exists $\varphi: P/V \longrightarrow U$ which is an R-isomorphism. Thus, $\delta \varphi \in Epi_R(P/V, P/B)$, so by (2) there exists $\alpha \in Hom_R(P, P/V)$ with $\delta \varphi \alpha = \lambda$. Now, let $h = \varphi \alpha$. Hence, $h \in Hom_R(P, U)$ with $\delta h = \delta \varphi \alpha = \lambda$.

(3) $\Longrightarrow$ (1). Clear.

Lemma (2.15): ([1], Prop. 1.25) An R- module H is directly finite iff $\beta \lambda = I$ implies that $\lambda \beta = I$ for each $\beta, \lambda \in End_R(H)$.

In the next results presented below discuss the relationships between YCPP-modules and some well-known modules such as, co-Hopfian, Hopfian and directly finite modules.

Proposition (2.16): Let P is an YCPP-module and FYC-module. Then P is a directly finite iff P is a Hopfian.

Proof: Assume that P is a directly finite. Let β be any R-epimorphism in $End_R(P)$. Since P is a FYC-module, then ker $(\beta) \subseteq_{yc} P$. Therefore, by Prop. (2.8) there exists $\lambda \in End_R(P)$ with $\beta \lambda = I$ where I is the identity map of P. But P is a directly finite, so $\lambda \beta = I$. Hence, λ is an R-automorphism. Conversely, let P be a Hopfian. Now, let $\beta, \lambda \in End_R(P)$ and $\beta \lambda = I$, we have β is an R-epimorphism. Hence, β is an R-automorphism since P is Hopfian. So $\lambda = \beta^{-1}$. Therefore, $\lambda \beta = I$.

Corollary (2.17): If P is an yc-projective and FYC-module. Then P is a directly finite iff P is a Hopfian.

Proposition (2.18):Let P be any YCPP-module and FYC-module. If P is a co-Hopfian, then it is Hopfian.

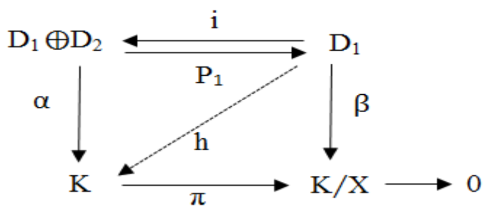
Proof**:** Let β be any R-epimorphism in $End_R(P)$ and let $I_P : P \rightarrow P$ be an identity map on P.

By y-closed-pseudo-projectivity of P there exists $\lambda \in End_R(P)$ such that $\beta \lambda = I_P$ which implies that λ is an R-monomorphism. As P is a co-Hopfian, λ is an R-automorphism. Thus, $\beta = \lambda^{-1}$ is an R-automorphism on P. Hence, P is a Hopfian

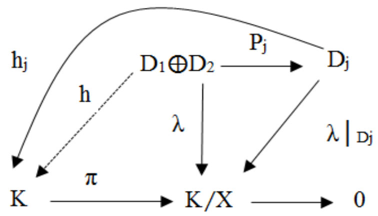
Now, we give some properties of direct sum in term YCP -K-projective modules.

Theorem (2.19): Let D_1 and D_2 be R-modules. Then $D_1 \oplus D_2$ is YCP-K-projective iff D_1 and D_2 are YCP-K-projective.

Proof: Assume that $D_1 \oplus D_2$ is an YCP-K-projective. To show that D_1 is an YCP-K-projective. Let $X \subseteq_{yc} K$ and let $\beta \in Epi_R(D_1, K/X)$. Consider the illustration below:



Where P_1 is the projection map and i is an inclusion map. Clearly, $\beta P_1 \in Epi_R(D_1 \oplus D_2, K/X)$. Since $D_1 \oplus D_2$ is an YCP-K-projective, there exists $\alpha \in Hom_R(D_1 \oplus D_2, K)$ such that $\pi \alpha = \beta P_1$. Now, let $h = \alpha i$. It follows that $\pi h = \pi \alpha i = \beta P_1 i = \beta I = \beta$. Therefore, D_1 is an YCP-K-projective. Similarly, we can show that D_2 is an YCP-K-projective. Conversely, suppose that D_1 and D_2 are YCP-K-projective modules. Let $\lambda \in Epi_R(D_1 \oplus D_2, K/X)$ with $X \subseteq_{yc} K$. Therefore, $\lambda|_{D_j}: D_j \longrightarrow K/X$, is an R-epimorphism, where j = 1, 2. Consider the illustration below:

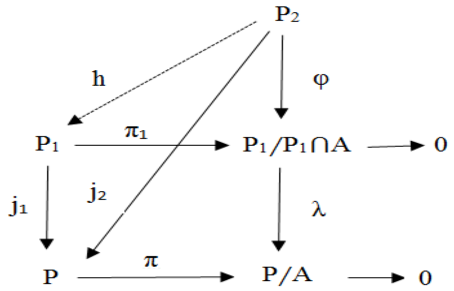


Since D_j is an YCP-K-projective, it follows that $\pi h_j = \lambda|_{D_j}$ for some $h_j \in Hom_R(D_j, K)$. Now, let $h = h_j P_j$. Hence, $h \in Hom_R(D_1 \oplus D_2, K)$ with $\pi h = \pi h_j P_j = \lambda|_{D_j} P_j = \lambda$.

Corollary (2.20): Any direct summand of YCP-K-projective module is an YCP-K projective.

Proposition (2.21): Let $P = P_1 \oplus P_2$ be an R-module. If P_2 is an YCP-P_1-projective, then for each y-closed submodule A of P with $P = P_1 + A$, there exists a submodule X of A such that $P = P_1 \oplus X$.

Proof: Assume that P_2 is an YCP-P_1- projective. Let $A \subseteq_{yc} P$ such that $P = P_1 + A$. If $n_2 \in P_2$ then $n_2 = n_1 + a$, where $n_1 \in P_1$, $a \in A$. Let $\varphi: P_2 \longrightarrow P_1/P_1 \cap A$ be a map defined by $\varphi(n_2) = \varphi(n_1 + a) = n_1 + P_1 \cap A$. To show that φ is well defined. If $n_2 = n_2^*$ in P_2 then $n_2 = n_1 + a$ and $n_2^* = n_1^* + a^*$, for some $n_1, n_1^* \in P_1$ and $a, a^* \in A$. Then $n_1 - n_1^* = a^* - a \in P_1 \cap A$. So, $\varphi(n_2) = \varphi(n_2^*)$. Clearly φ is an R- epimorphism. By the second isomorphism theorem $P_1/P_1 \cap A \cong P_1 + A/A = P/A$. So, exists $\lambda: P_1/P_1 \cap A \longrightarrow P/A$ is an R-isomorphism defined as follows $\lambda(n_1 + P_1 \cap A) = n_1 + A$, for all $n_1 \in P_1$. Consider the illustration below:



Where π_1 and π are the natural epimorphisms and j_1 and j_2 are the inclusion maps. Since P/A is a nonsingular and $P/A \cong P_1/P_1 \cap A$, we have $P_1/P_1 \cap A$ is a nonsingular, it follows that $P_1 \cap A \subseteq_{yc} P_1$. Since P_2 is an YCP-P_1-projective, there exist $h \in Hom_R(P_1, P_2)$ such that $\pi_1 h = \varphi$. Now, let $X = (j_1 h - j_2)(P_2)$, to show that $X \subseteq A$. Let $x \in X$, $x = (j_1 h - j_2)(n_2)$ where $n_2 \in P_2$ So, $x + A = (j_1 h - j_2)(n_2) + A = \pi((j_1 h - j_2)(n_2)) = \pi j_1 h(n_2) - \pi j_2(n_2) = \lambda \pi_1 h(n_2) - \pi j_2(n_2) = \lambda \varphi(n_2) - \pi j_2(n_2) = \lambda(n_1 + P_1 \cap A) - \pi(n_2) = n_1 + A - (n_2 + A) = n_1 - n_2 + A = -a + A = A$, hence $x \in A$ and so $X \subseteq A$. Clearly, $P = P_1 + P_2 = P_1 + X$. Now, let $y \in P_1 \cap X$ we get $y = (j_1 h - j_2)(n_2) = j_1 h(n_2) - j_2(n_2)$. So, $y = h(n_2) - n_2$. Thus $n_2 = h(n_2) - y \in P_1 \cap P_2 = 0$ we have $y = 0$. Hence, $P = P_1 \oplus X$.

In [2, Coro. (8.2.2)], was proved that; any R-module be projective iff each R-module be injective iff R be semi-simple iff any simple R-module is projective. The next theorem gives characterization of semisimple ring in terms of YCP-K-projective.

Proposition (2.22): Let R be a ring, then for all R-module K the next statement are equivalent

1) Any YCP-K-projective is K-projective.
2) The direct sum of each family of YCP-K-projective is a K-projective
3) The direct sum of each two YCP-K-projective is a K-projective
4) R is semisimple

Proof: 1) $\Longrightarrow$ 2) By Theorem. (2.19). (2) $\Longrightarrow$ (3) clear.

(3)$\Longrightarrow$ (4) let D is simple R-module, thus by remark 2.2, (4) we get D is YCP-K-projective. It follows that $D \oplus D$ is projective by (3). Therefore, D is a projective by [1, Proposition (4.32)]. Hence, R is semi-simple by [2, Coro. (8.2.2)].

(4) $\Longrightarrow$ (1) since any R-module is projective, then (1) hold.

3. Conclusion

Through this paper, we reached the following conclusions: Any pseudo-K-projective module is an YCP-K-projective, we give an example of an YCP-K-projective which is no pseudo-K-injective. Also, we have given the sufficient condition for equivalence; YCP-K-projective and pseudo-K-projective, K-projective. And the direct sum of YCP-K-projective is an YCP-K-projective. And any direct summand of YCP-K-projective is an YCP-K-projective.

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