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Gumbel Type -2 Stress – Strength P(X<Y<Z) n-Cascade Reliability Estimation

N. S. Karam[1] and H. A. Jasem[2]

[1,2] Department of Mathematics, College of Education, Mustansiriyah University Iraq
(dr_nadaskaram@uomustansiriyah.edu.iq[1],
haneenaseed20@uomustansiriyah.edu.iq)

A B S T R A C T

In this paper, is derived the reliability of n-cascade stress-strength system model based on the Gumbel Type- 2 Distribution (GT-2) with unknown parameter λ and known parameters β , for probability of n- components having strengths Y between two stresses X and Z .In the Gumbel Type- 2 Distribution ,there are six methods of parameters and this system reliability estimators where discussed by using the Maximum Likelihood, Moments Method, Least Square Method, Weighted Least Square Method , Regression and Percentile ,based estimators on the simulation technique , these estimates are compared by the mean square error criteria for both small, medium and large samples .It has been concluded that the maximum likelihood estimator performs better than all the options considered.

KeyWords: **:** n-Cascade system, Stress-strength Reliability, Probability P(X<Y<Z), The Gumbel Type- 2 Distribution, Estimation Methods.

1. Introduction

Pandit and Sriwastav first developed and studied the reliability of n-cascade system [1]. Let X and Y are random variables , the reliability of a component is defined as the probability that the strength Y working on it, is greater than stress X in stress strength model R=pr$(X \geq Y)$. We consider the case in which the component works only when the strength Y lies between two certain values of components stress (say X and Z).This means the strength lies within certain limits see [2 , 3 , 4 , 5].

Can be defined the reliability under this system as: $R = P(X < Y < Z)$

Let X, Y and Z are random variables ,where X and Z are the component stress ,and Y is the strength on this component .The reliability estimation under the assumption that the strength of a component Y lies in an interval between two stress (X_1 and X_2). i.e. $R = P(X_1 < Y < X_2)$.[4]

For $X \sim GT-2(\lambda\, ,\beta\,)$ can be rewrite [3]:

Gumbel Type -2 Stress – Strength

$$F(x) = e^{-\lambda x^{-\beta}} \qquad ; x > 0, \lambda, \beta > 0 \qquad (1)$$

And the probability density function (pdf):

$$\mathrm{f}(x) = \lambda \beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}} \qquad ; x > 0; \lambda, \beta > 0 \qquad (2)$$

Where λ and β are the shape parameters , since f(x) is probability density function, then we can rewrite equation (2) as:

$$\int_0^\infty x^{-(\beta+1)} e^{-\lambda x^{-\beta}} dx = \frac{1}{\lambda \beta}$$

The main idea of this paper is to reach the reliability of the n-cascade stress-strength system for The Gumbel Type- 2 where strength on the component is subjected to two stress in section 2. In order to find the estimators of the shape parameters $(\lambda, \lambda_1, \lambda_2)$for the three random variables, six different estimation methods (Maximum Likelihood, the Moment, Least Square, Weighted Least Square, Regression and Percentile Methods) are used and then the reliability parameter is estimated in section 3. A simulation study was conducted to compare the performance of the six different estimators of the reliability in section 4, based on nine experiments of shape parameter values and at different sample sizes of (15) for small, (25) for medium and (100) for large sample sizes. The comparison is made by the mean square error criteria (MSE), and the conclusions are discussed in section 5.

2. Reliability Formulation

The stress-strength model indicates the probability of a component strength falling between two stresses, and the reliability formula for the stress-strength models is given as follows [6]:

$$\begin{aligned} R &= P(X<Y<Z) \\ &= \int_0^\infty P(X < x, Z > x / Y = x) dF_x(x) \\ &= \int_0^\infty H_X(x)\big(1 - G_Z(x)\big) f(x) dx \qquad (3) \end{aligned}$$

The strength is adjusted by the factor (k) [7], when you get a failed in the cascade system .Let y_i ;i=1,2,…,n be the stress on n-components ,then $y_2 = ky_1, y_3 = ky_2 = k^2 y_1, \dots, y_i = k^{i-1} y_1$

Then we suppose that the components operate independently .

$$R_n = R(1) + R(2) + \cdots + R(n) \qquad (4)$$

Let x_i, z_i and y_i ; i=1,2,…,n are independent random variables with H(x),G(z) cumulative distribution functions (cdf) and $f(y_1)$ probability density function (pdf),then

$$\begin{aligned} R(1) &= P(x < y_1 < z) = P(y_1 > x) - P(y_1 > x, y_1 > z) \\ &= \int_0^\infty H_x(y_1) f(y_1) dy_1 - \int_0^\infty H_x(y_1) G_z(y_1) f(y_1) dy_1 \qquad (5) \end{aligned}$$

And

$$R(2) = P(x < y_1 < z)^c P(x < y_2 < z)$$

$$= [1 - R(1)][P\big((y_2 = ky_1) > x\big) - P(ky_1 > x, ky_1 > z)]$$

$$= [1 - R(1)][\int_0^\infty H_x(ky_1) f(y_1) dy_1 - \int_0^\infty H_x(ky_1) G_z(ky_1) f(y_1) dy_1] \quad (6)$$

Similarly

$$R(3) = [1 - R(1)] * [1 - R(2)]$$

$$[\int_0^\infty H_x(k^2 y_1) f(y_1) dy_1 - \int_0^\infty H_x (k^2 y_1) G_z(k^2 y_1) f(y_1) dy_1] \quad (7)$$

Generalization

$$R(m) = [1 - R(1)] * [1 - R(2)] * \dots * [1 - R(m-1)]$$

$$\int_0^\infty H_x(k^{m-1} y_1) f(y_1) dy_1 - \int_0^\infty H_x (k^{m-1} y_1) G_z(k^{m-1} y_1) f(y_1) dy_1] \quad (8)$$

Let Y be a random strength variable following GT-2(λ,β) with cumulative density function F(x) .And let X and Z be a random stress variables following GT-2(λ_1,β) and GT-2(λ_2,β)with cdf $H_X(x), G_z(z)$, respectively . Assumed that Y independent from X and Z, then:

$$H_X(x) = e^{-\lambda_1 x^{-\beta}} \qquad ;\ x > 0; \lambda_1, \beta > 0 \quad (9)$$

$$G_z(z) = e^{-\lambda_2 z^{-\beta}} \qquad ; z > 0; \lambda_2, \beta, > 0 \quad (10)$$

And let y_1 be GT-2 (λ,β) random variable strength with (pdf) give :

$$f(y_1) = \lambda\, \beta\, {y_1}^{-(\beta+1)}\, e^{-\lambda\, {y_1}^{-\beta}} \qquad ; y_1 > 0, \lambda, \beta \quad (11)$$

Since f (y_1) is pdf function.

Then from equation (5), we have $R(1) = A_1 - B_1$, Then

$$A_1 = \int_0^\infty H_X(y_1)\, f(y_1)\, dy_1$$

$$= \int_0^\infty e^{-\lambda_1 {y_1}^{-\beta}}\, \lambda\, \beta\, {y_1}^{-(\beta+1)}\, e^{-\lambda\, {y_1}^{-\beta}}\, dy_1$$

$$= \lambda\, \beta\, \int_0^\infty {y_1}^{-(\beta+1)}\, e^{-(\lambda+\lambda_1)\, {y_1}^{-\beta}}\, dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1}$$

$$B_1 = \int_0^\infty \mathrm{H_x}(y_1)\ \mathrm{G_z}(y_1)\ \mathrm{f}(y_1)\mathrm{d}y_1$$

$$= \int_0^\infty e^{-\lambda_1\, y_1{}^{-\beta}} e^{-\lambda_2\, y_1{}^{-\beta}} \quad \lambda\, \beta\, y_1{}^{-(\beta+1)}\ e^{-\lambda\, y_1{}^{-\beta}}\ dy_1$$

$$= \lambda\, \beta\ \int_0^\infty y_1{}^{-(\beta+1)}\ e^{-(\lambda+\lambda_1+\lambda_2)\, y_1{}^{-\beta}}\ dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1+\lambda_2}$$

Then ,we get

$$R(1) = A_1 - B_1$$

$$= \frac{\lambda}{\lambda+\lambda_1} - \frac{\lambda}{\lambda+\lambda_1+\lambda_2}$$

$$= \frac{\lambda\, \lambda_2}{(\lambda+\lambda_1)(\lambda+\lambda_1+\lambda_2)}$$

And we have

$$R(2) = [1 - R(1)][A_2 - B_2]$$

$$A_2 = \int_0^\infty H_X(ky_1)\ f(y_1)dy_1$$

$$= \int_0^\infty e^{-\lambda_1 (k\, y_1)^{-\beta}}\ \lambda\, \beta\, y_1{}^{-(\beta+1)}\ e^{-\lambda\, y_1{}^{-\beta}}\ dy_1$$

$$= \lambda\, \beta\ \int_0^\infty y_1{}^{-(\beta+1)}\ e^{-(\lambda+\lambda_1\, k^{-\beta})\, y_1{}^{-\beta}}\ dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1 k^{-\beta}}$$

$$B_2 = \int_0^\infty \mathrm{H_x}(ky_1)\ \mathrm{G_z}(\mathrm{k}y_1)\ \mathrm{f}(y_1)\mathrm{d}y_1$$

$$= \int_0^\infty e^{-\lambda_1 (k\, y_1)^{-\beta}} e^{-\lambda_2 (k\, y_1)^{-\beta}} \quad \lambda\, \beta\, y_1{}^{-(\beta+1)}\ e^{-\lambda\, y_1{}^{-\beta}}\ dy_1$$

$$= \lambda\, \beta\ \int_0^\infty y_1{}^{-(\beta+1)}\ e^{-(\lambda+\lambda_1 k^{-\beta}+k^{-\beta}\lambda_2)\, y_1{}^{-\beta}}\ dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1 k^{-\beta}+\lambda_2 k^{-\beta}}$$

So

$$R(2) = [1 - R(1)][\ \frac{\lambda}{\lambda+\lambda_1 k^{-\beta}} - \frac{\lambda}{\lambda+\lambda_1 k^{-\beta}+\lambda_2 k^{-\beta}}\ ]$$

$$= [1 - R(1)][\frac{\lambda\, \lambda_2\, k^{-\beta}}{(\lambda+\lambda_1 k^{-\beta})(\lambda+\lambda_1 k^{-\beta}+\lambda_2 k^{-\beta})}]$$

$$A_3 = \int_0^\infty H_X(k^2 y_1)\, f(y_1) dy_1$$

$$= \int_0^\infty e^{-\lambda_1 (k^2\, y_1)^{-\beta}}\, \lambda\, \beta\, {y_1}^{-(\beta+1)}\, e^{-\lambda\, {y_1}^{-\beta}}\, dy_1$$

$$= \lambda\, \beta\, \int_0^\infty {y_1}^{-(\beta+1)}\, e^{-(\lambda+\lambda_1\, k^{-2\beta})\, {y_1}^{-\beta}}\, dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1 k^{-2\beta}}$$

$$B_3 = \int_0^\infty H_X(k^2 y_1)\, G_Z(k^2 y_1)\, f(y_1) dy_1$$

$$= \int_0^\infty e^{-\lambda_1 (k^2\, y_1)^{-\beta}} e^{-\lambda_2 (k^2\, y_1)^{-\beta}}\ \ \lambda\, \beta\, {y_1}^{-(\beta+1)}\, e^{-\lambda\, {y_1}^{-\beta}}\, dy_1$$

$$= \lambda\, \beta\, \int_0^\infty {y_1}^{-(\beta+1)}\, e^{-(\lambda+\lambda_1 k^{-2\beta}+k^{-2\beta}\lambda_2)\, {y_1}^{-\beta}}\, dy_1$$

$$= \frac{\lambda}{\lambda+\lambda_1 k^{-2\beta}+\lambda_2 k^{-2\beta}}$$

$$R(3) = [1-R(1)][1-R(2)][A_3 - B_3]$$

$$= [1-R(1)][1-R(2)][\frac{\lambda\, \lambda_2\, k^{-2\beta}}{(\lambda+\lambda_1 k^{-2\beta})(\lambda+\lambda_1 k^{-2\beta}+\lambda_2 k^{-2\beta})}]$$

ln general

$$A_m = \frac{\lambda}{\lambda+\lambda_1 (k^{m-1})^{-\beta}}$$

$$B_m = \frac{\lambda}{\lambda+\lambda_1 (k^{m-1})^{-\beta}+\lambda_2 (k^{m-1})^{-\beta}}$$

$$R(m) = [1-R(1)][1-R(2)] \ldots [1-R(m)][A_m - B_m]$$

$$R_3 = R(1) + R(2) + R(3)$$

3.Estimation Method

In this section, to find the estimator of The Gumbel Type -2 unknown shape parameters $\lambda, \lambda_1, \lambda_2$ and Reliability R of the stress-strength model are used six different estimation methods. These methods are Maximum Likelihood, Moments Method, Least Square Method, Weighted Least Square Method, Regression Method and Percentile Method. These methods are used to arrive at the best reliability estimate.

3.1. Maximum Likelihood Estimator (MLE)

Let $y_1, y_2, \ldots y_n$ be strength random sample of size n from GT-2(λ,β) where λ is unknown parameter and β are known. The most widely method is maximum likelihood used for parameter estimation [8]. Then the MLE function is given by:

$$L(y_1, y_2, \ldots, y_n; \lambda, \beta) = (\lambda\beta)^n \prod_{i=1}^{n} {y_i}^{-(\beta+1)}\, e^{-\lambda \sum_{i=1}^{n} {y_i}^{-\beta}} \tag{12}$$

By taking the natural logarithm function of equation (12) we get:

$$\ln L = n\ln\lambda + n\ln\beta - (\beta+1)\sum_{i=1}^{n}\ln(y_i) - \lambda\sum_{i=1}^{n}(y_i)^{-\beta} \qquad (13)$$

By taking partial differential of equation (13) with respect to the unknown parameter λ, and equation the result to zero, we get:

$$\frac{\partial lnL}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}(y_i)^{-\beta} \rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n}(y_i)^{-\beta} = 0$$

$$\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n}(y_i)^{-\beta}}$$

ln the same way, let $x_1, x_2, \dots, x_m$ and $z_1, z_2, \dots, z_m$ are two random stress samples of size (m) from GT-2(λ_1,β) and GT-2(λ_2,β), respectively. Then the MLE estimators of the unknown parameters λ_1, λ_2 are:

$$\hat{\lambda}_{1MLE} = \frac{m}{\sum_{j=1}^{m}(x_j)^{-\beta}} \quad ; \quad \hat{\lambda}_{2MLE} = \frac{m}{\sum_{b=1}^{m}(z_b)^{-\beta}}$$

Then the MLE estimator of Reliability R is given by substitution $\hat{\lambda}_{MLE}$, $\hat{\lambda}_{1MLE}$ and $\hat{\lambda}_{2MLE}$ in equations (5) , (6) and (7) ,using the invariant property of this method as:

$$\hat{R}(1)_{MLE} = \frac{\hat{\lambda}_{MLE}\hat{\lambda}_{2MLE}}{(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE})(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE}+\hat{\lambda}_{2MLE})}$$

$$\hat{R}(2)_{MLE} = \left[1-\hat{R}(1)_{MLE}\right]\left[\frac{\hat{\lambda}_{MLE}\hat{\lambda}_{2MLE}k^{-\beta}}{(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE}\,k^{-\beta})(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE}\,k^{-\beta}+\hat{\lambda}_{2MLE}k^{-\beta})}\right]$$

$$\hat{R}(3)_{MLE} = \left[1-\hat{R}(1)_{MLE}\right]\left[1-\hat{R}(2)_{MLE}\right]\left[\frac{\hat{\lambda}_{MLE}\hat{\lambda}_{2MLE}k^{-2\beta}}{(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE}\,k^{-2\beta})(\hat{\lambda}_{MLE}+\hat{\lambda}_{1MLE}\,k^{-2\beta}+\hat{\lambda}_{2MLE}k^{-2\beta})}\right]$$

3.2. Moments Method (MOM)

ln 1894 Pearson introduced The Moment Method. To estimate the population parameter, this method was one of the methods used [3]. To derived the moment method estimators of parameters of GT-2. Then their population means are given by:

$$E(y) = \lambda^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right) \quad , E(x) = \lambda_1^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right) \quad , E(z) = \lambda_2^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta})$$

According to the moments method, equating the samples mean with the corresponding populations mean, then the moment estimators of $\lambda, \lambda_1, \lambda_2$ are :

$$\hat{\lambda}_{MOM} = \left(\frac{\bar{y}}{\Gamma(1-\frac{1}{\beta})}\right)^{\beta} \quad ; \quad \hat{\lambda}_{1MOM} = \left(\frac{\bar{x}}{\Gamma(1-\frac{1}{\beta})}\right)^{\beta} \quad ; \quad \hat{\lambda}_{2MOM} = \left(\frac{\bar{z}}{\Gamma(1-\frac{1}{\beta})}\right)^{\beta}$$

We can obtain the approximate estimator of R, by substitution $\hat{\lambda}_{MOM}$, $\hat{\lambda}_{1MOM}$ and $\hat{\lambda}_{2MOM}$ in equations (5) , (6) and (7) as bellow:

$$\hat{R}(1)_{MOM} = \frac{\hat{\lambda}_{MOM}\hat{\lambda}_{2MOM}}{(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM})(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM}+\hat{\lambda}_{2MOM})}$$

$$\hat{R}(2)_{MOM} = \left[1-\hat{R}(1)_{MOM}\right]\left[\frac{\hat{\lambda}_{MOM}\,\hat{\lambda}_{2MOM}\,k^{-\beta}}{(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM}\,k^{-\beta})(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM}\,k^{-\beta}+\hat{\lambda}_{2MOM}k^{-\beta})}\right]$$

$$\hat{R}(3)_{MOM} = \left[1-\hat{R}(1)_{MOM}\right]\left[1-\hat{R}(2)_{MOM}\right]$$

$$\left[\frac{\hat{\lambda}_{MOM}\lambda_{2MOM}k^{-2\beta}}{(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM}\,k^{-2\beta})(\hat{\lambda}_{MOM}+\hat{\lambda}_{1MOM}\,k^{-2\beta}+\hat{\lambda}_{2MOM}k^{-2\beta})}\right]$$

3.3. Least Square Method (LS)

In 1988 suggested Swain, Venkatraman and Wilson the Least Square Method to estimate the parameters of Beta distribution. By minimizing the sum between the value and it's expected value [5]. we obtained the least square estimators . Let $y_1, y_2, \dots, y_n$ be a random sample of the strength order statistics of size (n) from GT-2(λ,β). Now the LS can be written as :

$$S=\sum_{i=1}^{n}[F(y_i)-E(F(y_i))]^2$$
(14)

Where $E(F(y_i)) = P_i$ the plotting and $P_i=\frac{i}{n+1}$,i=1,2,…,n

ln equation (14) putting the cdf of GT-2, we get:

$$S=\sum_{i=1}^{n}[e^{-\lambda\, y_i^{-\beta}} - P_i]^2 \qquad (15)$$

So then,

$$S=\sum_{i=1}^{n}[-\lambda y_i^{-\beta} - q_i]^2 \qquad (16)$$

Where $q_i = \ln(F(y_i)) = \ln(P_i)$

By taking partial differential to equation (16) with respect to the unknown shape parameter λ and equating the result to zero, we will get:

$$\frac{\partial S}{\partial \lambda}=2\sum_{i=1}^{n}\left[-\lambda\, y_i^{-\beta} - q_i\right] * [-y_i^{-\beta}]$$

$$\hat{\lambda}\sum_{i=1}^{n} y_i^{-2\beta} + \sum_{i=1}^{n} q_i\, y_i^{-\beta}=0$$

$$\hat{\lambda}_{LS}=-\frac{\sum_{i=1}^{n} q_i\, y_i^{-\beta}}{\sum_{i=1}^{n} y_i^{-2\beta}}$$

In the same technique , the LS estimators for λ_1 *and* λ_2 , we can write by :

$\hat{\lambda}_{1LS}=-\frac{\sum_{j=1}^{m} q_j\, x_j{}^{-\beta}}{\sum_{j=1}^{m} x_j{}^{-2\beta}}$

$\hat{\lambda}_{2LS}=-\frac{\sum_{b=1}^{m} q_b\, z_b{}^{-\beta}}{\sum_{b=1}^{m} z_b{}^{-2\beta}}$

Where $P_j = \frac{j}{m+1}$, j=1,2,…,m and $P_b = \frac{b}{m+1}$, b=1,2,…,m

We obtain the LS estimator for the reliability R, by substitution $\hat{\lambda}_{LS}$, $\hat{\lambda}_{1LS}$ and $\hat{\lambda}_{2LS}$ in equations (5) , (6) and (7),will be approximately as:

$\hat{R}(1)_{LS} = \frac{\hat{\lambda}_{LS}\hat{\lambda}_{2LS}}{(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS})(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS}+\hat{\lambda}_{2LS})}$

$\hat{R}(2)_{LS} = \left[1-\hat{R}(1)_{LS}\right][\frac{\hat{\lambda}_{LS}\hat{\lambda}_{2LS}k^{-\beta}}{(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS}\,k^{-\beta})(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS}\,k^{-\beta}+\hat{\lambda}_{2LS}k^{-\beta})}]$

$\hat{R}(3)_{LS} = \left[1-\hat{R}(1)_{LS}\right]\left[1-\hat{R}(2)_{LS}\right][\frac{\hat{\lambda}_{LS}\hat{\lambda}_{2LS}k^{-2\beta}}{(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS}\,k^{-2\beta})(\hat{\lambda}_{LS}+\hat{\lambda}_{1LS}\,k^{-2\beta}+\hat{\lambda}_{2LS}k^{-2\beta})}]$

3.4. Weighted Least Square Method (WLS)

We can be obtained WLS estimator by minimizing the following equation :

$$\text{WS}=\sum_{i=1}^{n} w_i[F(y_i)-E\big(F(y_i)\big)]^2 \tag{17}$$

Where $w_i = \frac{1}{var[F(y_i)]} = \frac{(n+1)^2+(n+2)}{i(n-i+1)}$

The weighted least square estimator of the unknown shape parameter λ can be obtained by minimizing the following equation:

$$\text{WS}=\sum_{i=1}^{n} w_i[e^{-\lambda\, y_i{}^{-\beta}}-P_i]^2 \tag{18}$$

So then

$$\text{WS}=\sum_{i=1}^{n} w_i[-\lambda y_i{}^{-\beta}-q_i]^2 \tag{19}$$

By taking partial differential of equation (19) with respect to the unknown shape parameter λ and equating the result to zero, we will get:

$\frac{\partial wS}{\partial\lambda}=2\sum_{i=1}^{n} w_i\left[-\lambda\, y_i{}^{-\beta}-q_i\right]*[y_i{}^{-\beta}]$

$\hat{\lambda}\sum_{i=1}^{n} w_i\, y_i{}^{-2\beta}+\sum_{i=1}^{n} w_i q_i y_i{}^{-\beta} = 0$

$\hat{\lambda}_{WLS}=-\frac{\sum_{i=1}^{n} w_i q_i\, y_i{}^{-\beta}}{\sum_{i=1}^{n} w_i y_i{}^{-2\beta}}$

ln the same technique, the weighted lest square estimators for λ_1 and λ_2, are given by:

$$\hat{\lambda}_{1WLS}=-\frac{\sum_{j=1}^{m} w_j q_j \, x_j{}^{-\beta}}{\sum_{j=1}^{m} w_j x_j{}^{-2\beta}}$$

$$\hat{\lambda}_{2WLS}=-\frac{\sum_{b=1}^{m} w_b q_b \, z_b{}^{-\beta}}{\sum_{b=1}^{m} w_b z_b{}^{-2\beta}}$$

Where $w_j = \frac{1}{var[F(x_j)]} = \frac{(m+1)^2+(m+2)}{j(m-j+1)}$ and $w_b = \frac{1}{var[F(z_b)]} = \frac{(m+1)^2+(m+2)}{b(m-b+1)}$

We obtain the WLS estimator for the reliability R, by substitution $\hat{\lambda}_{WLS}$, $\hat{\lambda}_{1WLS}$ and $\hat{\lambda}_{2WLS}$ in equations (5) , (6) and (7), will be approximately as:

$$\hat{R}(1)_{WLS} = \frac{\hat{\lambda}_{WLS}\hat{\lambda}_{2WLS}}{(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS})(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS}+\hat{\lambda}_{2WLS})}$$

$$\hat{R}(2)_{WLS} = \left[1-\hat{R}(1)_{WLS}\right]\left[\frac{\hat{\lambda}_{WLS}\,\hat{\lambda}_{2WLS}\,k^{-\beta}}{(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS}\,k^{-\beta})(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS}\,k^{-\beta}+\hat{\lambda}_{2WLS}k^{-\beta})}\right]$$

$$\hat{R}(3)_{WLS} = \left[1-\hat{R}(1)_{WLS}\right]\left[1-\hat{R}(2)_{WLS}\right]\left[\frac{\hat{\lambda}_{WLS}\,\hat{\lambda}_{2WLS}k^{-2\beta}}{(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS}\,k^{-2\beta})(\hat{\lambda}_{WLS}+\hat{\lambda}_{1WLS}\,k^{-2\beta}+\hat{\lambda}_{2WLS}k^{-2\beta})}\right]$$

3.5. Regression Method (RG):

Regression is one of the important procedures that use auxiliary information to construct estimators with good efficiency

The standard regression equation [9]:

$$u_i = a + b\, v_i + \epsilon \qquad i = 1, 2, 3, \dots, n \tag{20}$$

Where u_i is dependent variable (response variable), v_i is independent variable (Explanatory Variable) and ϵ is the error r.v. independent identically. Let $y_1, y_2, \dots, y_n$ be a random strength sample of size (n) from GT-2 , then the GT-2 estimators of the unknown parameter λ can be obtained by taking the natural logarithm to equation (1) , as follows:

$$-\ln\big(F(y_i)\big) = \lambda y_i{}^{-\beta} \tag{21}$$

By putting the plotting position instead of $F(y_i)$ in equation (21) , we get :

$$-\ln(P_i) = \lambda y_i{}^{-\beta} \tag{22}$$

By comparison between equation (22) and equation (20) , we get:

$$u_i = -\ln(P_i)\,,\ \text{a=0}\quad ,\text{b} = \lambda\quad ,v_i = {y_i}^{-\beta} \tag{23}$$

Where b can be estimated by the minimizing summation of the squared error with respect to b ,then we get [9] :

$$\hat{b} = \frac{n\sum_{i=1}^{n} u_i v_i - \sum_{i=1}^{n} u_i \sum_{i=1}^{n} v_i}{n\sum_{i=1}^{n}(v_i)^2 - \left(\sum_{i=1}^{n} v_i\right)^2} \tag{24}$$

By substituting (23) in (24) ,the GT-2 estimator for the unknown parameter $\lambda\ is$:

$$\hat{\lambda}_{RG} = \frac{n\sum_{i=1}^{n} -\ln(P_i)\,{y_i}^{-\beta}\ - \sum_{i=1}^{n} -\ln(P_i)\sum_{i=1}^{n} {y_i}^{-\beta}}{n\sum_{i=1}^{n}(y_i)^{-2\beta} - \left(\sum_{i=1}^{n} {y_i}^{-\beta}\right)^2} \tag{25}$$

By same way ,we can get λ_1 , λ_2 as bellow:

$$\hat{\lambda}_{1RG} = \frac{m\sum_{j=1}^{m} -\ln(P_j)\,{x_j}^{-\beta}\ - \sum_{j=1}^{m} -\ln(P_j)\sum_{j=1}^{m} {x_j}^{-\beta}}{m\sum_{j=1}^{m}(x_j)^{-2\beta} - \left(\sum_{j=1}^{m} {x_j}^{-\beta}\right)^2} \tag{26}$$

$$\hat{\lambda}_{2RG} = \frac{m\sum_{j=1}^{m} -\ln(P_j)\,{z_j}^{-\beta}\ - \sum_{j=1}^{m} -\ln(P_j)\sum_{j=1}^{m} {z_j}^{-\beta}}{m\sum_{j=1}^{m}(z_j)^{-2\beta} - \left(\sum_{j=1}^{m} {z_j}^{-\beta}\right)^2} \tag{27}$$

the Regression Method estimator for the system reliability R is given by substituting the parameter estimators $\hat{\lambda}_{RG}$, $\hat{\lambda}_{1RG}$, $\hat{\lambda}_{2RG}$ at the equations (5) , (6) and (7) we get :

$$\hat{R}(1)_{RG} = \frac{\hat{\lambda}_{RG}\hat{\lambda}_{2RG}}{(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG})(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG}+\hat{\lambda}_{2RG})}$$

$$\hat{R}(2)_{RG} = \left[1-\hat{R}(1)_{RG}\right]\left[\frac{\hat{\lambda}_{RG}\hat{\lambda}_{2RG}k^{-\beta}}{(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG}\ k^{-\beta})(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG}\ k^{-\beta}+\hat{\lambda}_{2RG}k^{-\beta})}\right]$$

$$\hat{R}(3)_{RG} = \left[1-\hat{R}(1)_{RG}\right]\left[1-\hat{R}(2)_{RG}\right]\left[\frac{\hat{\lambda}_{RG}\hat{\lambda}_{2RG}k^{-2\beta}}{(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG}\ k^{-2\beta})(\hat{\lambda}_{RG}+\hat{\lambda}_{1RG}\ k^{-2\beta}+\hat{\lambda}_{2RG}k^{-2\beta})}\right]$$

3.6. Percentile Estimation:(PR)

The Percentile Estimator can be obtained by equating the sample percentile points with the population percentile points [10].

We have

$$F(y_1) = e^{-\lambda\, {y_1}^{-\beta}}$$

$$-\ln\big(F(y_1)\big) = \lambda\, {y_1}^{-\beta}$$

$$\frac{-\ln\big(\mathrm{F}(y_1)\big)}{\lambda} = {y_1}^{-\beta}$$

$$y_1 = \Big(-\frac{\ln\big(F(y_1)\big)}{\lambda}\Big)^{-\frac{1}{\beta}}$$

Since $P_i = \frac{i}{n+1}$; $i = 1, \dots, n$ is the most used estimator of $F(y_{(i)})$; $P_i = E\left(F(y_{(i)})\right)$, then λ can be obtained by minimizing:

$$\sum_{i=1}^{n} [y_{(i)} - (-\frac{\ln(P_i)}{\lambda})^{-\frac{1}{\beta}}]^2$$

$$\sum_{i=1}^{n} [y_{(i)} - (\frac{(-\ln(P_i))^{-\frac{1}{\beta}}}{\lambda^{-\frac{1}{\beta}}}]^2$$

$$\sum_{i=1}^{n} [y_{(i)} - \lambda^{\frac{1}{\beta}}(-\ln(P_i))^{-\frac{1}{\beta}}]^2 = P \qquad (28)$$

So , by taking partial differential of equation (28) with respect to the unknown shape parameter λ and equating the result to zero, we will get:

$$\frac{\partial P}{\partial \lambda} = 2\sum_{i=1}^{n} [\, y_{(i)} - \lambda^{\frac{1}{\beta}}(-\ln(P_i))^{-\frac{1}{\beta}}][\frac{1}{\beta}\, \lambda^{\frac{1}{\beta}-1}(-\ln(P_i))^{-\frac{1}{\beta}}]$$

$$2\sum_{i=1}^{n} [\, y_{(i)} - \lambda^{\frac{1}{\beta}}(-\ln(P_i))^{-\frac{1}{\beta}}][\frac{1}{\beta}\, \lambda^{\frac{1}{\beta}-1}(-\ln(P_i))^{-\frac{1}{\beta}}] = 0$$

$$\sum_{i=1}^{n} [\, y_{(i)} - \lambda^{\frac{1}{\beta}}(-\ln(P_i))^{-\frac{1}{\beta}}][\, \lambda^{\frac{1}{\beta}-1}(-\ln(P_i))^{-\frac{1}{\beta}}] = 0$$

$$\sum_{i=1}^{n} [\, y_{(i)}\, \lambda^{\frac{1}{\beta}-1}(-\ln(P_i))^{-\frac{1}{\beta}} - \lambda^{\frac{2}{\beta}-1}(-\ln(P_i))^{-\frac{2}{\beta}}] = 0$$

$$\sum_{i=1}^{n} [\, y_{(i)}(-\ln(P_i))^{-\frac{1}{\beta}} - \lambda^{\frac{1}{\beta}}(-\ln(P_i))^{-\frac{2}{\beta}}] = 0$$

$$\sum_{i=1}^{n} [\, y_{(i)}(-\ln(P_i))^{-\frac{1}{\beta}}] - \lambda^{\frac{1}{\beta}} \sum_{i=1}^{n} (-\ln(P_i))^{-\frac{2}{\beta}} = 0$$

$$\lambda^{\frac{1}{\beta}} = \frac{\sum_{i=1}^{n} y_{(i)}(-\ln(P_i))^{-\frac{1}{\beta}}}{\sum_{i=1}^{n} (-\ln(P_i))^{-\frac{2}{\beta}}}$$

$$\hat{\lambda} = [\frac{\sum_{i=1}^{n} y_{(i)}(-\ln(P_i))^{-\frac{1}{\beta}}}{\sum_{i=1}^{n} (-\ln(P_i))^{-\frac{2}{\beta}}}]^{\beta} \qquad (28)$$

In the same technique, the Percentile estimators for λ_1 and λ_2, are given by:

$$\hat{\lambda}_1 = [\frac{\sum_{j=1}^{m} x_{(i)}(-\ln(P_j))^{-\frac{1}{\beta}}}{\sum_{j=1}^{m} (-\ln(P_j))^{-\frac{2}{\beta}}}]^{\beta}$$

$$\hat{\lambda}_2 = [\frac{\sum_{j=1}^{m} z_{(j)}(-\ln(P_j))^{-\frac{1}{\beta}}}{\sum_{j=1}^{m} (-\ln(P_j))^{-\frac{2}{\beta}}}]^{\beta}$$

by substituting the parameter estimators at the equations (5) , (6) and (7) ,we get:

$$\hat{R}(1)_{PR} = \frac{\hat{\lambda}_{PR}\hat{\lambda}_{2PR}}{(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR})(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR}+\hat{\lambda}_{2PR})}$$

$$\hat{R}(2)_{PR} = \left[1-\hat{R}(1)_{PR}\right]\left[\frac{\hat{\lambda}_{PR}\hat{\lambda}_{2PR}k^{-\beta}}{(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR}\,k^{-\beta})(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR}\,k^{-\beta}+\hat{\lambda}_{2PR}k^{-\beta})}\right]$$

$$\hat{R}(3)_{PR} = \left[1-\hat{R}(1)_{PR}\right]\left[1-\hat{R}(2)_{PR}\right]\left[\frac{\hat{\lambda}_{PR}\hat{\lambda}_{2PR}k^{-2\beta}}{(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR}\,k^{-2\beta})(\hat{\lambda}_{PR}+\hat{\lambda}_{1PR}\,k^{-2\beta}+\hat{\lambda}_{2PR}k^{-2\beta})}\right]$$

4. Experimental Study

A experimental study in this section is used to determine the best estimate of the reliability with unknown parameters of The GT-2 , and performance of the six different estimates from the MLE, MOM, LS, WLS,RG and PR methods, are evaluated by using the mean square error criteria (MSE), with different sample sizes (15,25,100) and ($\beta = 2,3,2.5$) , for three different tables in each case of the parameter value β .

A experimental study is conducted for the nine different experiments, by using MATLAB 2018 to compare the performance of the reliability estimators by the following steps:

Step 1: Using the inverse function, random values of the random variables are generated by the following formula:$x = [-\ln(F(x))/\lambda]^{-1/\beta}$

Step 2: By using MSE criteria, the comparison of estimation methods is done:

MSE=$\frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$,where L=1000 is the number of replication in each experiment.

In the tables from 1 to 3, the results are recorded. Based on the MSE values, the performance of these estimators is compared. As MLE, MOM, LS, WLS,RG and PR sample sizes increasing with the value of MSE decreases in all three tables experiments, the best value of MSE in the experiment 1, experiment 2,experiment 3, experiment 4, experiment 5, experiment 6 , experiment 7 , experiment 8, experiment 9 is MLE estimator , followed by LS,RG, WLS,MOM,PR then the best value of MSE in the table 1,table 2 and table 3 is MLE estimator , followed by LS,RG, WLS,MOM ,PR. Therefore, regardless of whether the parameter values are equal or not , the MLE estimator is better than the LS,RG,WLS,MOM ,PR, through the small values of the MSE .

5.Conclusion

In this paper, we derived the reliability expression for n-cascade system of the stress-strength model and estimate it by using four methods for estimating reliability $P(X < Y < Z)$ where X, Y, Z follow GT-2 with different parameters. The results of the experiment confirm that MLE estimator is better than the estimators of MOM, LS, WLS,RG and PR for all sample sizes and for all experiments.

6.Acknowledgments

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Table 1: MSE value for Reliability estimators when $\beta = 2$

n,n1	MLE	MOM	LS	WLS	RG	PR
Experiment 1: $\lambda = 1.2\ ,\lambda_1 = 1.2\ ,\lambda_2 = 0.8\ ,R = 0.1773$						
15,25	0.0021	0.0091	0.0027	0.0036	0.0025	0.0186
25,100	0.0006	0.0039	0.0008	0.0015	0.0007	0.0138
15,100	0.0006	0.0037	0.0009	0.0016	0.0009	0.0116
15,15	0.0031	0.0115	0.0036	0.0044	0.0038	0.0200
25,25	0.0021	0.0099	0.0026	0.0036	0.0025	0.0199
100,100	0.0005	0.0036	0.0006	0.0014	0.0006	0.0124
Experiment 2: $\lambda = 0.9\ ,\lambda_1 = 0.8, \lambda_2 = 0.5\ ,R = 0.1757$						
15,25	0.0020	0.0101	0.0023	0.0031	0.0025	0.0201
25,100	0.0005	0.0032	0.0007	0.0015	0.0006	0.0108
15,100	0.0006	0.0035	0.0008	0.0015	0.0008	0.0124
15,15	0.0033	0.0122	0.0039	0.0047	0.0040	0.0206
25,25	0.0018	0.0095	0.0022	0.0030	0.0023	0.0189
100,100	0.0004	0.0036	0.0005	0.0012	0.0005	0.0127
Experiment 3: $\lambda = 1, \lambda_1 = 0.5, \lambda_2 = 0.7, R = 0.3302$						
15,25	0.0037	0.0149	0.0044	0.0057	0.0045	0.0280
25,100	0.0010	0.0073	0.0014	0.0028	0.0013	0.0221
15,100	0.0012	0.0069	0.0015	0.0028	0.0017	0.0209
15,15	0.0066	0.0228	0.0077	0.0090	0.0078	0.0370
25,25	0.0037	0.0157	0.0044	0.0061	0.0045	0.0288
100,100	0.0009	0.0063	0.0011	0.0026	0.0010	0.0209

Table 2: MSE value for Reliability estimators when $\beta = 3$

n,n1	MLE	MOM	LS	WLS	RG	PR
Experiment 4: $\lambda = 2, \lambda_1 = 1.9, \lambda_2 = 1.5, R = 0.1670$						
15,25	0.0015	0.0046	0.0018	0.0025	0.0018	0.0089
25,100	0.0004	0.0015	0.0005	0.0011	0.0006	0.0039
15,100	0.0005	0.0017	0.0006	0.0012	0.0007	0.0041
15,15	0.0025	0.0070	0.0029	0.0035	0.0031	0.0110
25,25	0.0016	0.0049	0.0020	0.0027	0.0019	0.0085
100,100	0.0004	0.0013	0.0005	0.0011	0.0004	0.0035
Experiment 5: $\lambda = 1.8, \lambda_1 = 0.9, \lambda_2 = 1.2, R = 0.2575$						

15,25	0.0024	0.0070	0.0028	0.0037	0.0029	0.0122
25,100	0.0006	0.0022	0.0007	0.0016	0.0007	0.0060
15,100	0.0006	0.0022	0.0008	0.0016	0.0008	0.0058
15,15	0.0036	0.0111	0.0044	0.0053	0.0045	0.0172
25,25	0.0024	0.0077	0.0028	0.0037	0.0029	0.0133
100,100	0.0006	0.0020	0.0007	0.0015	0.0006	0.0056
Experiment 6: $\lambda = 1.2, \lambda_1 = 1.5, \lambda_2 = 0.9, R = 0.1279$						
15,25	0.0012	0.0031	0.0014	0.0019	0.0014	0.0054
25,100	0.0003	0.0010	0.0004	0.0008	0.0004	0.0028
15,100	0.0004	0.0011	0.0005	0.0009	0.0005	0.0027
15,15	0.0018	0.0052	0.0022	0.0026	0.0022	0.0083
25,25	0.0012	0.0037	0.0014	0.0019	0.0015	0.0065
100,100	0.0003	0.0011	0.0004	0.0009	0.0004	0.0028

Table 3: MSE value for Reliability estimators when $\beta = 2.5$

Experiment 7: $\lambda = 0.5, \lambda_1 = 1.2, \lambda_2 = 1.7, R = 0.1690$						
n,n1	MLE	MOM	LS	WLS	RG	PR
15,25	0.0022	0.0073	0.0027	0.0037	0.0025	0.0133
25,100	0.0008	0.0031	0.0009	0.0017	0.0009	0.0081
15,100	0.0010	0.0039	0.0012	0.0020	0.0013	0.0094
15,15	0.0035	0.0102	0.0043	0.0052	0.0042	0.0164
25,25	0.0020	0.0067	0.0024	0.0033	0.0025	0.0121
100,100	0.0005	0.0022	0.0006	0.0014	0.0005	0.0071
Experiment 8: $\lambda = 0.3, \lambda_1 = 1.4, \lambda_2 = 0.9, R = 0.0696$						
15,25	0.0007	0.0028	0.0008	0.0011	0.0008	0.0061
25,100	0.0002	0.0011	0.0003	0.0005	0.0003	0.0036
15,100	0.0003	0.0011	0.0003	0.0005	0.0003	0.0032
15,15	0.0010	0.0039	0.0013	0.0016	0.0013	0.0070
25,25	0.0006	0.0020	0.0007	0.0011	0.0010	0.0041
100,100	0.0001	0.0007	0.0002	0.0004	0.0002	0.0025
Experiment 9: $\lambda = 2.5, \lambda_1 = 1.5, \lambda_2 = 0.5, R = 0.1030$						
15,25	0.0007	0.0032	0.0009	0.0013	0.0009	0.0071
25,100	0.0002	0.0009	0.0002	0.0005	0.0002	0.0037
15,100	0.0002	0.0009	0.0002	0.0006	0.0002	0.0036
15,15	0.0014	0.0054	0.0017	0.0020	0.0016	0.0098
25,25	0.0007	0.0037	0.0009	0.0013	0.0008	0.0080
100,100	0.0002	0.0009	0.0002	0.0005	0.0002	0.0038

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