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RESEARCH ARTICLE - MATHEMATICS

Small Annihilator Essential Submodules

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Abstract

In this research, we will present the concept of small annihilator essential submodules. It was generalization of essential submodules. In addition, investigate properties and characterizations of small annihilator essential submodules. Additionally, we introduce the notion of small annihilator closed submodules and small annihilator uniform modules which serve as generalization of closed submodules and uniform modules, respectively. These concepts extend existing ideas and offer new perspectives on the relationships between different submodule and module types within R-modules M.

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Introduction

In this research, all modules are unitary left modules and let R be a commutative ring with identity. A proper submodule D of an R-module M is referred to be essential and denoted by $(D \subseteq^{e} M)$ if for all $H \subseteq M$ where H$\neq$ 0 we have $D \cap H \neq 0$ [1]. Let M be an R-module and $D \subseteq M$ since $D \neq 0, D$ is referred to be annihilator essential submodule (briefly ann-essential) of M if $D \cap H = 0$ then $ann(H) \subseteq^{e} R \, for \, all \, H \subseteq M$. Equivalently: Let $0 \neq D \subseteq M, D$ is referred to be annihilator essential submodule of M if $D \cap H \neq 0, for \, all \, H \subseteq M$ such that $ann\,(H)$ is not essential in R [2]. Essential submodules have been extensively studied and generalized by various authors see ([3], [4], [5]). A submodule D of M is referred to be small in M denoted by $D \subseteq^{s} M$ if for some $H \subseteq M \, with \, D + H = M$ then $H = M$ [1],[6]. A module M is referred to be hollow module, if every proper submodule in M is small in M [1],[7],[8]. If $D \subseteq M$ and D has no proper essential submodule of an R-module M then D is referred to be closed submodule and denoted by $(D \subseteq^{c} M)$[9],[10],[11]. Similarly, If D has no proper ann-essential in M, it is called annihilator closed (briefly ann-closed) denoted by $(D \subseteq^{a.c} M)$ [2]. The set $Z(M) = \{m \in M : ann(m) \subseteq^{e} R\}$ where $ann(m) = \{r \in R : rm = 0\}$ is the singular of an R-module M, if $Z(M) = M$ then M is referred to be singular module, if $Z(M) = 0$ then M is referred to be non-singular module see[11],[12],[13] A module M is referred to be uniform modules if for all $D \subseteq M \, where \, D \neq (0) \, and \, D \subseteq^{e} M$[14],[15],[16]. A module M is referred to be annihilator uniform module (briefly ann-uniform) if $for \, all \, D \subseteq M \, where \, D \neq 0 \, and \, D \subseteq^{a.e} M$ [2]. A submodule D of an R-module M is referred to be fully invariant denoted by $(D \subseteq^{f.i} M)$ if every endomorphism $\alpha : M \longrightarrow M, \alpha(D) \subseteq D$[17],[18]. There are many generalizations of the ring theory in general see ([19],[20],[21]).

This paper is divided into three main sections, In the first section, we will present the concept of small annihilator essential submodules. Which generalizes essential submodules. We define a submodule D as a small annihilator essential submodule (briefly SAE-submodule) of M denoted by $D \subseteq^{s.a.e} M \, if \, D \cap H = 0$ implies that $ann(H) \subseteq^{e} R, for \, all \, H \subseteq^{s} M$. In addition, investigate properties and characterizations of small annihilator essential submodules. In the second section we introduced a new concept, which is one of the generalizations of closed submodules, called small annihilator closed submodules, where D is referred to be small annihilator closed submodule of M (briefly SAC-submodule) denoted by $D \subseteq^{s.a.c} M$ if D has no proper SAE-submodule in the module M. In the third section we introduced the concept of small annihilator uniform modules which is one of the generalizations of uniform modules since an R-module M is referred to be small annihilator uniform module (briefly SAU-module) if for all $D \subseteq M \, where \, D \neq 0 \, with \, D \subseteq^{s.a.e} M$. And examines their properties and relationships with other module concepts.

Throughout, various properties, characterizations, and examples of these concepts are explored to deepen the understanding of the structure of submodules in R-modules M.

1.Small annihilator essential submodules.

Definition(1.1): Let M be an R-module and $(0) \neq D \subseteq M$.$then\ D$ is referred to be small annihilator essential submodule of M (briefly SAE-submodule) denoted by $D \subseteq^{s.a.e} M\ if\ D \cap H = 0$ implies $ann(H) \subseteq^{e} R, for\ all\ H \subseteq^{s} M.$

Equivalently; Let $0 \neq D \subseteq M.then\ D$ is SAE-submodule if $D \cap H \neq 0$, $for\ all\ H \subseteq^{s} M$ such that $ann(H) \nsubseteq^{e} R.$

Examples and Remarks (1.2):

1. Every essential submodule of an R-module M is SAE-submodule.

Proof: Let $D \subseteq^{e} M$ such that $D \cap H = 0$ where $H \subseteq^{s} M$, since D is essential so $H = 0\ and\ ann0 = \{r \in R: r.0 = 0\} = R \subseteq^{e} R$. Hence, $D \subseteq^{s.a.e} M.$

But the opposite is not true. As, $M = Z_6$ as Z-module, both submodules $(2)\ and\ (3)$ with $(2) \cap A = 0\ and\ (3) \cap A = 0$ where $A = 0$ and 0 is small with $ann0 = Z \subseteq^{e} Z$ hence (2) and (3) are SAE-submodules in M which it is not essential.

2. Every ann-essential submodule is SAE-submodule.

But the opposite is not true. As, $M = Z_6\ as\ Z_6$-module. $(\bar{2}) \subseteq^{s.a.e} M,$ but $(\bar{2}) \nsubseteq^{a.e} M$ since $(\bar{2}) \cap (\bar{3}) = 0\ and\ ann(\bar{3}) = (\bar{2})$ which is not essential of Z_6.

3. Let M be an R-module, then $M \subseteq^{s.a.e} M.$
4. Let D be a submodule of a hollow R-module $M.$ If R is semi-simple ring then D is ann-essential submodule if and only if D is SAE-submodule of $M.$

Proof: ⇒ Clear from (2).

⇐ Let $D \subseteq^{s.a.e} M$ and let $H \subseteq M$ such that $D \cap H = 0$ since M is hollow and since D is SAE-submodule of M so $ann(H) \subseteq^{e} R$. But R is semi-simple ring. Thus, $ann(H) = R$ so $H = 0$ hence $D \subseteq^{a.e} M.$

5. Let D be a submodule of a hollow R-module $M.$ If R is semi-simple ring then D is essential submodule if and only if D is SAE-submodule of $M.$

Proof: ⇒ Clear from (1)

⇐ Let $D \subseteq^{s.a.e} M$ since R is semi-simple ring then from (4) we have $D \subseteq^{a.e} M.$ Also by [2, remarks and examples (2.1.2) (6)] we get $D \subseteq^{e} M$

6. If M is singular R-module then all its submodules are SAE-submodules.

Proof: Let $D \subseteq M$. To prove that $D \subseteq^{s.a.e} M$. Let $H \subseteq^{s} M$ such that $D \cap H = 0$. But M is singular thus, $ann(H) \subseteq^{e} R$. Then $D \subseteq^{s.a.e} M.$

But the opposite is not true where we not that $D \subseteq^{s.a.e} M$ for every submodule D in M while, M is non-singular module. For example, $M = Z\ as\ Z$-module.

Example(1.3): Let $M = Z_{42}$ as Z-module. All the submodules of M is SAE-submodule since:

Z_{42}	Essential	Ann-essential	SAE-submodule
Z_{42}	✓	✓	✓
$(\bar{2})$	×	✓	✓
$(\bar{3})$	×	✓	✓
$(\bar{6})$	×	✓	✓
$(\bar{7})$	×	✓	✓
$(\overline{14})$	×	✓	✓

$(\overline{21})$	×	✓	✓
$(\bar{0})$	×	✓	✓

The next results give the properties of SAE-submodule.

Remark (1.4):[22]

Let D and H be a submodules of an R-module M:

1. If $D \subseteq H$ $then$ $ann(H) \subseteq ann(D)$.
2. If D is a submodule of M and $\propto : M \longrightarrow M_1$ is a homomorphism then $ann(D) \subseteq ann(\propto (D))$.
3. $ann(D + H) = ann(D) \cap ann(H)$.
4. Let $\propto : M \longrightarrow M_1$ be an R-monomorphism and let D be a submodule of M then $ann(D) = ann(\propto (D))$.

From remark (1.4), we have the next generalization.

Proposition(1.5): Let both M and N be an R-modules and $\alpha: M \longrightarrow N$ be a monomorphism. If $D \subseteq^{s.a.e} N$ then $\propto^{-1} (D) \subseteq^{s.a.e} M$.

Proof: Let $H \subseteq^{s} M$ and $\propto^{-1} (D) \cap H = 0$ so $\alpha(\propto^{-1}(D) \cap H) = 0$ then $D \cap \alpha(H) = 0$, since D is SAE-submodule of N so $ann(\alpha(H)) \subseteq^{e} R$. since α is monomorphism so by remark (1.4) we get $ann(H) = ann(\alpha(H))$. Hence $ann(H) \subseteq^{e} R$.

Proposition(1.6): Let M and N be an R-modules and $\alpha: M \longrightarrow N$ is an isomorphism. If $D \subseteq^{s.a.e} M$ then $\alpha(D) \subseteq^{s.a.e} N$.

proof: Let $H \subseteq^{s} N$ such that $\alpha(D) \cap H = 0$. Where α is monomorphism we get $D \cap \propto^{-1} (H) = 0$. D is SAE-submodule of M so $ann(\propto^{-1} (H)) \subseteq^{e} R$, $ann(\propto^{-1} (H)) \subseteq ann(\alpha(\propto^{-1} (H))) \subseteq R$ by remark (1.4) since α is epimorphism so we get $ann(\propto^{-1} (H)) \subseteq ann(H) \subseteq R$. Where $ann(\propto^{-1} (H)) \subseteq^{e} R$ so we get $ann(H) \subseteq^{e} R$. Then $\alpha(D) \subseteq^{s.a.e} N$.

Proposition(1.7): Let M be an R-module and $D, H \subseteq M$ such that $D \subseteq H \subseteq M$. If $D \subseteq^{s.a.e} M$ then $H \subseteq^{s.a.e} M$.

Proof: Let $W \subseteq^{s} M$ such that $H \cap W = 0$ where $D \subseteq H$ so $D \cap W = 0$, since $D \subseteq^{s.a.e} M$ so $ann(W) \subseteq^{e} R$. Hence $H \subseteq^{s.a.e} M$.

The contrast cannot be considered valid. As, M=Z_{36} as Z-module and let $(\overline{18}) \subseteq (\bar{6}) \subseteq Z_{36}$, $(\bar{6}) \subseteq^{s.a.e} Z_{36}$ but $(\overline{18}) \nsubseteq^{s.a.e} Z_{36}$. Since $(\overline{18}) \cap (\overline{12}) \neq (0)$ and $ann(\overline{12}) = (\bar{3}) \subseteq^{e} Z_{36}$.

Corollary(1.8): Let D_1, D_2 are submodules of R-module M. If $D_1 \cap D_2 \subseteq^{s.a.e} M$ then both of $D_1 and$ D_2 are SAE-submodules of M.

Proof: Since $D_1 \cap D_2 \subseteq D_1 \subseteq M$ and $D_1 \cap D_2 \subseteq D_2 \subseteq M$. So by proposition (1.7) we have $D_1 \subseteq^{s.a.e} M$ and $D_2 \subseteq^{s.a.e} M$.

Corollary(1.9): If D, H are SAE-submodules of an R-module M then $D + H$ is SAE-submodule of M.

Proof: Since $D \subseteq D + H \subseteq M$ and $D \subseteq^{s.a.e} M$. Then by proposition(1.7) we get $D + H \subseteq^{s.a.e} M$.

Proposition(1.10): Let M is an R-module. If $D \subseteq^{e} W \subseteq M$ and $D_1 \subseteq^{s.a.e} W_1 \subseteq$M then $D \cap D_1 \subseteq^{s.a.e} W \cap W_1$.

Proof: Let $H \subseteq^{s} (W \cap W_1)$ such that $(D \cap D_1) \cap H = 0$ so $D \cap (D_1 \cap H) = 0$ since $D_1 \cap H \subseteq H \subseteq W$ and D is essential submodule of W. So $D_1 \cap H = 0$ with $H \subseteq^{s} W_1$ and D_1 is SAE-submodule of W_1 we get $ann(H) \subseteq^{e} R$. Hence $D \cap D_1 \subseteq^{s.a.e} W \cap W_1$.

The following proposition demonstrates that the direct sum of SAE-submodules is itself an SAE-submodule.

Proposition(1.11): Let $D_1 \subseteq M_1 \subseteq M$ and $D_2 \subseteq M_2 \subseteq M$ and M=$M_1 \oplus M_2$. Then $D_1 \oplus D_2 \subseteq^{s.a.e} M_1 \oplus M_2$ if and only if $D_1 \subseteq^{s.a.e} M_1$ *and* $D_2 \subseteq^{s.a.e} M_2$.

Proof: ⇐Let $H \subseteq^{s} M$ such that $H = H_1 \oplus H_2$ with $H_1 \subseteq^{s} M_1$ and $H_2 \subseteq^{s} M_2$, let $D \cap H = 0$ so $D_1 \cap H_1=0$ *and* $D_2 \cap H_2 = 0$ since $D_1 \subseteq^{s.a.e} M_1$ *and* $D_2 \subseteq^{s.a.e} M_2$ we have $ann(H_1)$ *and* $ann(H_2)$ are essential submodules of R. Hence $D = D_1 \oplus D_2 \subseteq^{s.a.e} M = M_1 \oplus M_2$.

⇒ Let $H=H_1 \oplus H_2 \subseteq^{s} M$ with $H_1 \subseteq^{s} M_1$ *and* $H_2 \subseteq^{s} M_2$ such that $D_1 \cap H_1=0$ *and* $D_2 \cap H_2=0$. Then $(D_1 \oplus D_2) \cap (H_1 \oplus H_2)=0$ since $D_1 \oplus D_2$ is SAE-submodule of $M_1 \oplus M_2$ so $ann(H_1 + H_2) \subseteq^{e} R$. By [22] we get $ann(H_1 + H_2)= ann(H_1) \cap ann(H_2)$ then $ann(H_1) \cap ann(H_2)$ is essential submodule of R. Since $ann(H_1) \cap ann(H_2) \subseteq ann(H_1) \subseteq R$ so $ann(H_1) \subseteq^{e} R$ and since $ann(H_1) \cap ann(H_2) \subseteq ann(H_2) \subseteq R$ so $ann(H_2) \subseteq^{e} R$.Thus $D_1 \subseteq^{s.a.e} M_1$ and $D_2 \subseteq^{s.a.e} M_2$.

Definition(1.12): An R-homomorphism $\alpha{:}M \to N$ where M and N be an R-modules is referred to be small annihilator essential homomorphism (briefly SAE-homomorphism) if α(M) $\subseteq^{\boldsymbol{s.a.e}} N$.

Proposition(1.13): Let M be an R-module and $D \subseteq M$. Then $D \subseteq^{s.a.e} M$ iff the inclusion function $i{:}D \to M$ is SAE-monomorphism.

Proof: Let i be SAE-monomorphism so $Im(i)$ is SAE-submodule of M. Since $m(i) = i(D) = D\ so$ $D \subseteq^{s.a.e} M$.

Conversely, since i is R-monomorphism and $i\ (D) = D$ and $D \subseteq^{s.a.e} M$. Hence i is SAE-monomorphism.

Example(1.14): Let M= $Z_8 \oplus Z_2$ as Z-module. Then all the submodules of M are:

N_1={$(\bar{1},\bar{0})$,$(\bar{2},\bar{0})$, $(\bar{3},\bar{0})$, $(\bar{4},\bar{0})$, $(\bar{5},\bar{0})$,$(\bar{6},\bar{0})$,$(\bar{7},\bar{0})$,$(\bar{0},\bar{0})$}.

N_2={$(\bar{2},\bar{0})$, $(\bar{4},\bar{0})$, $(\bar{6},\bar{0})$, $(\bar{0},\bar{0})$}.

N_3={$(\bar{4},\bar{0})$, $(\bar{0},\bar{0})$}.

N_4={$(\bar{0},\bar{1})$, $(\bar{0},\bar{0})$}.

N_5={$(\bar{1},\bar{1})$, $(\bar{2},\bar{0})$, $(\bar{3},\bar{1})$, $(\bar{4},\bar{0})$, $(\bar{5},\bar{1})$, $(\bar{6},\bar{0})$, $(\bar{7},\bar{1})$, $(\bar{0},\bar{0})$}.

N_6={$(\bar{2},\bar{1})$, $(\bar{4},\bar{0})$, $(\bar{6},\bar{1})$, $(\bar{0},\bar{0})$}.

N_7={$(\bar{4},\bar{1})$, $(\bar{0},\bar{0})$}.

N_8={$(\bar{2},\bar{0})$, $(\bar{4},\bar{0})$, $(\bar{6},\bar{0})$, $(\bar{2},\bar{1})$, $(\bar{4},\bar{1})$, $(\bar{6},\bar{1})$, $(\bar{0},\bar{1})$, $(\bar{0},\bar{0})$}.

N_9={$(\bar{4},\bar{0})$, $(\bar{4},\bar{1})$, $(\bar{0},\bar{1})$, $(\bar{0},\bar{0})$}.

N_{10}={$(\bar{0},\bar{0})$}.

N_{11}={$(\bar{1},\bar{0})$,$(\bar{2},\bar{0})$, $(\bar{3},\bar{0})$, $(\bar{4},\bar{0})$, $(\bar{5},\bar{0})$,$(\bar{6},\bar{0})$,$(\bar{7},\bar{0})$,$(\bar{0},\bar{0})$, $(\bar{0},\bar{1})$,

$(\bar{1},\bar{1})$, $(\bar{2},\bar{1})$, $(\bar{3},\bar{1})$, $(\bar{4},\bar{1})$, $(\bar{5},\bar{1})$, $(\bar{6},\bar{1})$, $(\bar{7},\bar{1})$}.

$Z_8 \oplus Z_2$	essential	ann-essential	SAE-submodules
N_1	×	✓	✓
N_2	×	✓	✓
N_3	×	✓	✓
N_4	×	✓	✓
N_5	×	✓	✓
N_6	×	✓	✓
N_7	×	✓	✓
N_8	✓	✓	✓
N_9	✓	✓	✓
N_{10}	×	✓	✓
N_{11}	✓	✓	✓

Definition (1.15):

Let M be a non-zero R-module is referred to be small-totally essential module (briefly STE-module) if $for\ all (0) \neq D \subseteq^{s.a.e} M\ then\ D \subseteq^{e} M$.

Remarks and Examples(1.16):

1. $Z_4\ as\ Z$-module is STE-module.
2. Every uniform module is STE-module.
3. $Z_6\ as\ Z$-module is not STE-module.

Example(1.17): Let $M = Z_{30}\ as\ Z$-module.

Z_{30}	Essential	Ann-essential	SAE-submodules
Z_{30}	✓	✓	✓
$(\bar{2})$	×	×	×
$(\bar{3})$	×	×	×
$(\bar{5})$	×	×	×
$(\bar{6})$	×	×	×
$(\overline{10})$	×	×	×
$(\overline{15})$	×	×	×
$(\bar{0})$	×	×	×

2. Small annihilator closed submodules:

Definition(2.1): Let M be an R-module and $D \subseteq M$. If D has no proper SAE-submodule in M then D is referred to be small annihilator closed submodule (briefly SAC-submodule) denoted by $D \subseteq^{s.a.c} M$. That is if $\exists\ H \subseteq M$ such that $D \subseteq^{s.a.e} H \subseteq M$ implies that $D = H$.

Remarks and Examples(2.2):

1. Every SAC-submodule is closed.

Proof: Let $D \subseteq^{s.a.c} M$ and let $H \subseteq M$ such that $D \subseteq^{e} H \subseteq M$. By examples and remarks(1.2)(1) we get $D \subseteq^{s.a.e} H \subseteq M$. But $D \subseteq^{s.a.c} M$ so $D = H$. Hence $D \subseteq^{c} M$.

The reverse side is not true in general; if we take $M = Z_6$ *as* Z-module, we get $(\bar{3}) \subseteq^{c} M$ while $(\bar{3}) \nsubseteq^{s.a.c} M$ since $(\bar{3}) \subseteq^{s.a.e} M$.

2. Every SAC-submodule is annihilator-closed submodule.

Proof: Let $D \subseteq^{s.a.c} M$ and let $H \subseteq M$ such that $D \subseteq^{a.e} H \subseteq M$ by examples and remarks(1.2)(2) we get $D \subseteq^{s.a.e} H \subseteq M$. But $D \subseteq^{s.a.c} M$ so $D = H$. Hence D is ann-closed submodule in M.

3. The intersection of SAC-submodules of an R-module M is also an SAC-submodule.

Proof: Let $D \subseteq^{s.a.c} M$ *and* $H \subseteq^{s.a.c} M$ and let $L \subseteq M$ with $D \cap H \subseteq^{s.a.e} L \subseteq M$ by corollary (1.8) then $D \subseteq^{s.a.e} L \subseteq M$ *and* $H \subseteq^{s.a.e} L \subseteq M$. Since $D \subseteq^{s.a.c} M$ and $H \subseteq^{s.a.c} M$ *so* $D = L = H$ then $D \cap H = L$. Thus $D \cap H$ is SAC-submodule in M.

4. Let M be an R-module and $D \subseteq M$. If $D \subseteq^{s.a.c} M$ *and* $D \subseteq^{s.a.e} M$ *then* $D = M$.
5. Let D is submodule of an R-module M. If R is semi-simple ring then D is SAC-submodule if and only if D is closed submodule.

Proof: ⇒ Clear by (1).

⇐Let $D \subseteq^{c} M$ and $D \subseteq^{s.a.e} H \subseteq M$. Since R is semi-simple ring so by remarks and examples(1.2)(5) we get $D \subseteq^{e} H \subseteq M$. But $D \subseteq^{c} M$ so D =H. Hence $D \subseteq^{s.a.c} M$.

Proposition(2.3): Let M be an R-module and $D, H \subseteq M$ such that D$\subseteq H \subseteq M$. If $D \subseteq^{s.a.c} M$ *then* $D \subseteq^{s.a.c} H$.

Proof: Assume that D is SAE-submodule of W, since $W \subseteq H$ so $W \subseteq M$. But $D \subseteq^{s.a.c} M$.Hence $D = W$.

Corollary(2.4): Let M be an R-module and $D, H \subseteq M$. If $D \cap H \subseteq^{s.a.c} M$ *then* $D \cap H$ is SAC-submodule in both of D and H.

Corollary(2.5): Let M be an R-module and $D, H \subseteq^{s.a.c} M$ then both of D and H are SAC-submodules in $D + H$.

Proof: Since $D \subseteq D + H \subseteq M$ and H$\subseteq D + H \subseteq M$ so by proposition(2.3) we have D and H are SAC-submodules in $D + H$.

Proposition(2.6): Let $\alpha: M_1 \to M_2$ be a monomorphism and let $D \subseteq M_1$. If $D \subseteq^{s.a.c} M_1$ then $\alpha(D) \subseteq^{s.a.c} M_2$.

Proof: assumption that $\alpha(D)$ is SAE-submodule of H where $H \subseteq M_2$. By proposition(1.5) $\alpha^{-1}(\alpha(D)) \subseteq^{s.a.e} \alpha^{-1}(H)$ then $D \subseteq^{s.a.e} \alpha^{-1}(H) \subseteq M_1$. But $D \subseteq^{s.a.c} M_1$then $D = \alpha^{-1}(H)$ so $\alpha(D) = H$. Hence $\alpha(D) \subseteq^{s.a.c} M_2$.

Proposition(2.7): If every submodule of an R-module M is SAC-submodule then every submodule of M is a direct summand.

Proof: Where every submodule of M is SAC-submodule so by remarks and examples(2.2)(1) we have every submodule is closed and by [1] we have every submodule is direct summand.

The contrast cannot be considered valid. As, $M = \mathrm{Z}_8 \oplus \mathrm{Z}_2$ *as* Z-module, we get $< (\bar{1}, \bar{0}) >$ is direct summand but not SAC-submodule.

Proposition(2.8): Let M be a chained R-module and $D, H \subseteq M$. If $D \subseteq^{s.a.c} H$ *and* $H \subseteq^{s.a.c} M$ *then* $D \subseteq^{s.a.c} M$.

Proof: Let $A \subseteq M$ with $\subseteq^{s.a.e} A \subseteq M$. Where M is chained module so we get two cases: If A be a submodule of H, since $D \subseteq^{s.a.c} H$ so $D = A$ thus $D \subseteq^{s.a.c} M$. If $H \subseteq A$, since D is SAE-submodule of A so by proposition(1.7) we have H is SAE-submodule of A. But $H \subseteq^{s.a.c} M$ *so* $H = A$ then D is SAE-submodule of H. Since $D \subseteq^{s.a.c} H$ then $D = H$. Thus $D \subseteq^{s.a.c} M$.

Proposition(2.9): Let M be a hollow R-module and let $D \subseteq H \subseteq M$ with D is SAC-submodule in M. If $H \subseteq^{s.a.e}$M *hen* $\frac{H}{D} \subseteq^{s.a.e} \frac{M}{D}$.

Proof: Let $\frac{A}{D} \subseteq^{s} \frac{M}{D}$ $such\ that\ A \subseteq^{s} M\ with\ \frac{H}{D} \cap \frac{A}{D} = 0\ \ then\ H \cap A = D$. Where $H \subseteq^{s.a.e} M$ and A is essential submodule of A so by proposition(1.10) we have $H \cap A \subseteq^{s.a.e} M \cap A$ so $D \subseteq^{s.a.e} A$ with $A \subseteq M$. But $D \subseteq^{s.a.c} M$ so $D = A$ and thus $\frac{A}{D} = 0$. Then $ann(\frac{A}{D}) = ann(0) = R$, and R is essential of R. Hence $\frac{H}{D} \subseteq^{s.a.e} \frac{M}{D}$.

Proposition(2.10): Let M be an R-module and $D \subseteq^{c} M\ where\ D \neq (0)$.If every SAE-submodule of D is small-totally essential submodule of M then $D \subseteq^{s.a.c} M$.

Proof: Let $D \subseteq^{c} M\ where\ D \neq (0)$ and $H \subseteq M$ such that $\subseteq^{s.a.e} H \subseteq M$. By hypothesis H is STE-submodule so $D \subseteq^{e} H$. But $D \subseteq^{c} M$, then $D = H$. Thus $D \subseteq^{s.a.c} M$.

Remark(2.11): Let $D \subseteq M$. If M is STE-module then $(0) \neq D \subseteq^{c} M$ if and only if D is SAC-submodule.

Proof:$\Rightarrow$ Let $D \subseteq^{c} M\ where\ D \neq (0)$ such that D is SAE-submodule of H since $H \subseteq M$ where M is STE-module then D is essential of H. But $D \subseteq^{c} M$ so $D = H$, Thus $D \subseteq^{s.a.c} M$.

$\Leftarrow$ Clear by [remarks and examples (2.2)(1)].

Proposition(2.12): Let M=$M_1 \oplus M_2$ be an R-module where M_1, M_2 are submodules of M. Let D and H be non-zero submodules of M_1 and M_2 respectively. If $D \oplus H$ is SAC-submodule in $M_1 \oplus M_2$ then $D \subseteq^{s.a.c} M_1$ and $\subseteq^{s.a.c} M_2$.

Proof: assumption that D is SAE-submodule of $A_1$where A_1a submodule of M_1 and H is SAE-submodule of A_2 where A_2 a submodule of M_2 so by proposition(1.11) we have D$\oplus H \subseteq^{s.a.e} A_1 \oplus A_2 \subseteq M$. But $D \oplus H$ is SAC-submodule of M then $D \oplus H = A_1 \oplus A_2$.Thus $D = A_1$ and H=A_2 then D is SAC-submodule of M_1 and H is SAC-submodule of M_2.

Proposition(2.13): Let M=$M_1 \oplus M_2$ by an R-module where $M_1$and M_2 are submodules of M and let $D \subseteq^{s.a.c} M_1\ and\ H \subseteq^{s.a.c} M_2$. Then $D \oplus H \subseteq^{s.a.c} M_1 \oplus M_2$.

Proof: Suppose that $D \oplus H \subseteq^{s.a.e} A$ where $A \subseteq M$. so by proposition(1.11) $D \subseteq^{s.a.e} A_1\ and\ H \subseteq^{s.a.e} A_2$ where $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$. But $D \subseteq^{s.a.c} M_1\ and\ H \subseteq^{s.a.c} M_2\ so\ D = A_1\ and\ H = A_2$ that is $\oplus\ H = A_1 \oplus A_2$. Thus $D \oplus H \subseteq^{s.a.c} M_1 \oplus M_2$.

3. Small annihilator uniform modules:

Definition(3.1): A non-zero R-module M is referred to be small annihilator uniform module (briefly SAU-module) if $for\ all\ D \subseteq M\ where\ D \neq (0), D \subseteq^{s.a.e} M$.

Remarks and Examples(3.2):

1. Let $M = Z_{42}\ as\ Z$-module. M is SAU-module since $for\ all\ D \subseteq M, D \subseteq^{s.a.e} M$.
2. Let $M = Z_{30}\ as\ Z$-module. M is not SAU-module since every submodule of M is not SAE-submodule.
3. Every simple R-module is SAU-module.

The contrast cannot be considered valid. As, $M = Z_{12}\ as\ Z$-module. M is SAU-module but non-simple.

4. Every uniform module is SAU-module.

The contrast cannot be considered valid. As, $M = Z_{12}\ as\ Z$-module. M is SAU-module but not uniform since $(\bar{3}) \cap (\bar{4})$= 0.

5. Every ann-uniform module is SAU-module.

The contrast cannot be considered valid. As, $M = Z_6\ as\ Z_6$-module. M is not ann-uniform but M is SAU-module where $(\bar{2})$, $(\bar{3}) \subseteq M$ but $(\bar{2})$, $(\bar{3}) \subseteq^{s.a.e} M$.

6. Every submodule of SAU-module is SAU-module.

Proof: Let M is SAU-module and $D \subseteq M$. Let $0 \neq H \subseteq D$ then $H \subseteq M$. Where M is SAU-module so $H \subseteq^{s.a.e} M$ let $H \cap A$=(0) where $A \subseteq^{s} D$ so $A \subseteq^{s} M$ since $H \subseteq^{s.a.e} M$ so $ann(A) \subseteq^{e} R$ then $H \subseteq^{s.a.e} D$. Then every submodule of D is SAE-submodule hence D is SAU-module.

Proposition(3.3): Let M be a hollow R-module and R is semi-simple ring. Then M is uniform module if and only if M is SAU-module.

Proof: ⇒ Clear by (remarks and examples(3.2)(4)).

⇐ Let M is SAU-module so $for\ all\ D \subseteq M, D \subseteq^{s.a.e} M$. But R is semi-simple ring then by remarks and examples(1.2)(5) we get $for\ all\ D \subseteq M, D \subseteq^{e} M$. Hence M is uniform module.

Proposition(3.4): Let M be a hollow SAU-module and let $D \subseteq^{s.a.c} M$. Then $\frac{M}{D}$ is SAU-module.

Proof: Let $0 \neq \frac{H}{D} \subseteq \frac{M}{D}$ thus $0 \neq H \subseteq M$. But M is SAU-module then $H \subseteq^{s.a.e} M$. Since $D \subseteq^{s.a.c} M$ by proposition(2.9) then $\frac{H}{D} \subseteq^{s.a.e} \frac{M}{D}$. Thus $\frac{M}{D}$ is SAU-module.

Proposition(3.5): Let M=$M_1 \oplus M_2$ is a distributive module and $D \subseteq M$. If M_1 and M_2 are SAU-modules then M is SAU-module.

Proof: Let $D \subseteq M$ where $D \neq (0)$ since M is distributive module so $D = D \cap M = (D \cap M_1) \oplus (D \cap M_2)$. Where $0 \neq (D \cap M_1) \subseteq M_1$ and $0 \neq (D \cap M_2) \subseteq M_2$ then $(D \cap M_1) \subseteq^{s.a.e} M_1$ and $(D \cap M_2) \subseteq^{s.a.e} M_2$ then by proposition(1.11) $D \subseteq^{s.a.e} M$. Thus M is SAU-module.

Recall that: Referred a module M to be duo module if $for\ all\ D \subseteq M, D \subseteq^{f.i} M$. [23]

Proposition(3.6): Let $M = M_1 \oplus M_2$ be a duo module and $D \subseteq M$. If M_1 and M_2 are SAU-modules then M is SAU-module.

Proof: Let $0 \neq D \subseteq M$ where M is duo-module so $D \subseteq^{f.i} M$ and thus $D = D \cap M = (D \cap M_1) \oplus (D \cap M_2)$. Where $0 \neq (D \cap M_1) \subseteq M_1$ and $0 \neq (D \cap M_2) \subseteq M_2$ then $(D \cap M_1) \subseteq^{s.a.e} M_1$ and $(D \cap M_2) \subseteq^{s.a.e} M_2$ then by proposition(1.11) $D \subseteq^{s.a.e} M$. Hence M is SAU-module.

Proposition(3.7): Let α:$M \longrightarrow N$ be an R-isomorphism. If M is SAU-module then N is an SAU-module.

Proof: Let α:$M \longrightarrow N$ is an isomorphism and M be an SAU-module. Let $0 \neq D \subseteq N$ so $\alpha^{-1}(D) \subseteq M$. Where M is SAU-module so $\alpha^{-1}(D) \subseteq^{s.a.e} M$ by proposition(1.5) $\alpha(\alpha^{-1}(D)) = D$ is SAE-submodule of N. Hence N is SAU-module.

Proposition(3.8): Let α:$M \longrightarrow N$ is an R-monomorphism. If N is SAU-module then M is SAU-module.

Proof: Let $0 \neq D \subseteq M\ then\ \alpha(D) \neq 0\ .If\ \alpha(D) = 0\ so\ D \subseteq ker(\alpha) = 0$ which is contradiction. Since N is SAU-module so $\alpha(D)$ is SAE-submodule of N then by proposition(1.5) $D \subseteq^{s.a.e} M$. Thus M is SAU-module.

Proposition(3.9): If $M_1$and M_2 are SAU-modules then $M_1 \cap M_2$ is SAU-module.

Proof: Let $D = M_1 \cap M_2$. Since $D = M_1 \cap M_2 \subseteq M_1$ and $D = M_1 \cap M_2 \subseteq M_2$ and since M_1 and M_2 are SAU-modules then by remarks and examples(3.2)(6) we have D is SAU-module.

4. Conclusions.

In this work, we draw the following conclusions: D is SAE-submodule in M if D is essential. D is SAE-submodule in M if D is ann-essential. Every submodule of M is SAE-submodule if M is singular. The direct sum of SAE-submodules is SAE-submodule. If $D_1 \cap D_2$ is SAE-submodule in M then both of D_1 and D_2 are SAE-submodules. Also, D is closed in M if D is SAC-submodule. D is ann-closed if D is SAC-submodule. The intersection of SAC-submodules is SAC-submodule. If D is SAE-submodule and SAC-submodule then D = M. The direct sum of SAC-submodules is SAC-submodule. Additionally, M is SAU-module if M is uniform. M is SAU-module if M is ann-uniform. M is SAU-module if M is simple.

Every submodule of SAU-module is SAU-module. The direct sum of SAU-module is SAU-module if *M* is distributive module. The intersection of SAU-modules is SAU-module.

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