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RESEARCH ARTICLE - MATHEMATICS

On Approximation Properties of Unbounded Functions via Bernstein Polynomials in Weighted Space

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Abstract

In this work, we introduced some main result of approximation of unbounded function by probabilistic tools on $[0,1]$ and approximate unbounded function by polynomial Bernstein of derivative.

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***Keywords*:** unbounded function, weighted $L_{p,\beta}$ space, Bernstein polynomial derivative, direct theorem approximation, inverse theorem

1-Introduction

Uniform approximation is a mathematical concept that involves approximating certain functions by other functions, such that a certain accuracy is achieved at all points in the domain. The concept is widely used in mathematical analysis, function theory, and computer science. In fields like signal analysis, where simple or polynomial functions can be utilized to approximate more complicated functions, the concept of uniform approximation is especially crucial. A uniform approximation is considered "homogeneous" if the asymptotic function and the original function diverge by an equally small amount at every point in the domain. The approximation theory , which claims that any One can approximate a continuous function on a closed interval. by polynomial functions, is one of the primary uses of uniform approximation. One of the foundational findings in mathematical analysis is the "Weierstrass Approximation Theorem." Uniform approximation is generally a useful technique for simplifying and analyzing complex functions, which facilitates the study of their characteristics and behavior. The most significant and fascinating concrete operators in the continuous function space. are Bernstein polynomials. In 1983, Swetits and wood [1] presented a research on how the integral modulus of smoothness is used to measure the degree of approximation in L_p spaces by positive linear operators. It is demonstrated that the class of functions with a second derivative that belong to L_p does not achieve the conjectured ideal degree of approximation. In 1989, Quak and Ewald [2] reported a study that used The average modulus of smoothness, as it is known, also known as the τ-modulus of first orders, to provide The approximation error's estimates L_p-norm $\|f - L_f\|_p$. In 1991 Yuankwei and Shunsheng [3] used the modified Lupas operator to explore the rate of Functions of bounded variation approximation in 1991. They derive approximation theories and make what they believe to be the most accurate estimate. In 1996 [4] Swetits, introduced s study about the degree of approximation in L_p-spaces by positive linear operators in estimated in terms of the integral modulus of smoothness. It is showed that the conjectured optimal degree of approximation is not attained in the class of functions having a second derivative belong to L_p.In 2000 [5] used characterisation theorems for the optimal approximation in linear 2-normed space were established by Elumalai and Souruparani. Additionally, give some findings regarding the 2-norm's right-hand Gâteaux derivative. Included are some findings about the operators $\pi_{G,Z}$ and the functionals $e^{G,z}$ .in 2012 [6] in $\mathrm{Lp(Td)}, \mathrm{p} \geq 1$. A dell et al. developed operators in 2014 [7] for utilizing algebraic polynomials in L_p-spaces to approximate Riemann integrable functions on [0,1]. In 2018 [8], Koc and Akgun researched two-sided approximation with averaged modulus of smoothness by trigonometric polynomials, and obtained direct and inverse theorems. In 2021 The direct and converse algebraic approximation theorems were established by Auad and Fayyadh [9]. in 2023 with reference to the modulus of smoothness of fractional order in weighted spaces and the best approximations of bounded functions. The best approximation by a number of linear operators, such as the Bernstein-Durrmeyer operator, the Fejer operator, the Jackson operator, and the properties of the same theorem between the modulus of smoothness and the K-functional, were examined by Auad [10], [11]. In 2024 Raheam A Al-Saphory, Abdullah A Al-Hayani, Alaa A Auad Presented concept of best one-sided approximation of unbounded functions in weighted space by using algebraic operators in connection of average modulus for smoothness type. More precisely, we demonstrate the existing of some algebraic operators for single-directional approximation of unbounded functions in the terms of average modulus of smoothness. We also show that, the direct theorems to approximate unbounded functions in weighted space by one-sided approximation methods [12] In 2024 Alaa Adnan and Mohammed hilal in the study of classical functional analysis under a neutrosophic environment to handle indeterminate and inconsistent information. Where the neutrosophic norm function assigns to each vector in the linear space a neutrosophic number, which is a number with a truth, indeterminacy, and falsity component. The main aim of this work is to study and discuss the important properties of proximinality of specific sets and new results for a large class in neutrosophic normed space [13]. In 2025 Enas Atalla and Alaa Adnan presented approximation unbounded mappings in equiponderant space by linear operators and this approximation in terms modulus of smoothness.[14] In 2025 Alaa Adnan and Mohammed hilal In this presented focus on some famous mathematical spaces like $\mathrm{L}_{\delta},\mathrm{p}(\mathrm{U})$ when we work on displaying a feature the immediate and contrary theorems of unrestrained functions in the space $\mathrm{L}_{\delta},\mathrm{p}(\mathrm{U})$ are considered. Also, some characteristics of modification symmetric and modulus of neutrosophic smoothness have been discussed. Moreover, the identical among approximate tools such as the neutrosophic K-functional and neutrosophic modulus of smoothness [15] .

Suppose that g is a real function that is defined on the [0,1] interval. Assume E[X]be the expected value of the random variable X, and Assume $\mathrm{A}_{n,x}$ be a binomial random variable with parameters n, x. The reader is advised to consult the following references [20,21,22,23,24,25].

The definition of the Bernstein polynomials is:

$$B_n g(x) = \sum_{i=0}^{n} \binom{n}{x} x^i (1-x)^{n-i} g\left(\frac{i}{n}\right) = Eg\left(\frac{A_{n,x}}{n}\right) \qquad (1.1)$$

In this paper, we presented definition of Bernstein polynomial. For used of approximate of unbounded functions in weighted space. Also, the direct approximation theorem of unbounded function via modulus of smoothness considered and the Expend function approximate of unbounded function in terms small Constants. Also we proved some main result by using Modulus of smoothness defined by

$$w_k(g,\delta)_{p,\beta} = \left\|\Delta_\delta^k(g,.)\right\|_{p,\beta}$$

$$\Delta_\delta^k(g,t) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k g(t + \frac{rh}{2} - kh)$$

The degree of approximation of unbounded function defined by

$$E_n(g,\delta)_{p,\beta} = \inf \{\|g - p_n\|_{p,\beta} \qquad p_n \in \Pi_n\}$$

Where Π_n the space of all algebraic or trigonometric with degree less than or equal n.

To show our primary result, we now need certain characteristics of Bernstein's polynomials.

The following characteristics of Bernstein polynomials are generally met [10].

1. Bernstein polynomials are increasing and monotone.
2. $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$
3. $\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n}$
4. For $\delta > 0, and \left|\frac{k}{n} - x\right| \geq \delta$, implies $\sum_{k=0}^{n} \binom{n}{k} x^k (1-k)^{n-k} \leq \frac{1}{4n\sigma}$

Theorem 1.1 (Weierstrass Approximation Theorem)

Suppose that $g : [0,1] \to \mathbb{R}$ be a function that is continuous $B_n(g,.): [0,1] \to \mathbb{R}$

belong to the Bernstein polynomial, which as defined by (1.1). Consequently, as well as each $\epsilon > 0$, There, are $n \in N$ so that is

$$|g(t) - B_n(g,t)| < \epsilon \quad \text{for every } t \in [0,1].$$

There are numerous authors who study the limits of this approximate, including
Right up until 1997 When g is the Lipschitz function, meaning that there is a constant L, both Gzyl and Palacios ([9]) examined the unboundedness of the approximation.
so that $|g(x) - g(y)| \leq L|x - y|$
For any $x, y \in [0,1]$. Then yield the rate $\frac{\ln n}{n}$ as described below:

$$|g(t) - B_n(g,t)| \leq C\frac{\ln n}{n} \quad \text{for all} \quad t \in [0,1].$$

When dealing with multi-dimension, K. Neammanee and S. Sirisub ([3]) approximated using probalistic techniques and shown that for $\epsilon > 0$,

$$\|g(t) - B_n(g,t)\| \leq \epsilon \quad \text{for everyone} \quad t \in [0,1]. \qquad (1.2)$$

In (1.3) K.Neammanee and S.Siribsub, they did not give unbounded of the approximation.
K. Neammanee ([15]) provided an unbounded estimate in 2001. when discussing the Lipschitz function. He proved that

$$\|g(t) - B_n(g,t)\| \leq C\frac{\ln n}{n} \quad for\ t \in [0,1].$$

Assume $g : R \to \mathbb{R}$ be holder function with exponent $\beta > 0$, i.e, there exists a constant C such

$$\text{that } \|g(t) - B_n(g,t)\| \leq C\|x - y\|^{\beta} \ for\ each\ \ x, y \in [0,1].$$

Then there exist a constant C depend on g such that

$$\|g(t) - B_n(g,t)\| \leq \frac{C}{n^{\frac{\beta}{2}}} \quad for \quad t \in [0,1] \ \ and\ n \in \mathbb{N}. \qquad (1.3)$$

The condition of Punkla and Sontonsinsongvon will be relieved to an unbounded function in this work. Through the modulus of smoothness, we will demonstrate how the Bernstein polynomial directly approximates unbounded functions in weighted space.

Theorem 1.2. [1] Let $g : [0,1] \to \mathbb{R}$ be unbounded function , then there exists a constant $C > 0$ so that

$$E_n(g,\delta)_{p,\beta} \leq \|g(t) - B_n(g,t)\| \leq C\omega(n^{-1}) \quad for\ t \in [0,1]$$

Where the modulus $\omega(\delta)$ of g is defined for $\delta > 0$ by

$$\omega(g,\delta) = \sup_{\substack{t_1,t_2 \in [0,1] \\ \|t_1 - t_2\| \leq \delta}} \|g(t_1) - g(t_2)\|.$$

Note that the modulus of g depends of δ, g and the interval[0,1]. So that $\omega(\delta)$ is shorthand
For $\omega(g; [0,1]; \delta)$. It g is holder with exponent $\beta > 0$, we 've $\omega(\frac{1}{n}) = \frac{C}{n^{\frac{\beta}{2}}}$.

Hence theorem 1.2 generizes (1.3).

2. of Main Results

In this part, we presented some results of direct approximation theorems by Bernstein polynomial and its derivative to approximate unbounded functions in weighted spaces

Theorem 2.1. Let $g \in L_{p,\beta}([0,1])$, $1 \leq p < \infty$, and for $\delta > 0$. Then

$$E_n(g,\delta)_{p,\beta} \leq C(p)\ \ \omega_k(g,\delta)_{p,\beta}$$

Proof:

$$\|g(x) - B_n(g,x)\|$$

$$B_n(g,x) = \sum_{i=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} g(\frac{i}{n})$$

$$\|g - B_n(g,t)\|_{p,\beta}$$

$$= \left\| g - \sum_{i=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} g\left(\frac{i}{n}\right) \right\|_{p,\beta}$$

$$= \left\| \sum_{i=0}^{n} \binom{n}{j} x^{j}(1-x)^{n-j} \left(g - g\left(\frac{i}{n}\right)\right) \right\|_{p,\beta}$$

$$= \int_{X} \left(\left| \sum_{i=0}^{n} \binom{n}{j} x^{j}(1-x)^{n-j}\; g(x) - g\left(\frac{i}{n}\right) \right|^{p} \; dx \right)^{\frac{1}{p}}$$

$$= \int_{X} \left(\left|\Delta_{n}^{\delta}\left(g(x)\right)\right|^{p} dx\right)^{\frac{1}{p}} \qquad \text{s.t } X = [0,1]$$

$$\leq \sup \left\| \Delta_{n}^{\delta}(g,.) \right\|_{p,\beta}$$

$$\leq c\omega_{n}\left(g, \frac{1}{n}\right)_{p,\beta}.$$

3 Converge of $(B_n g)(x)'$ in direction of $g'(x)$.

In the last section, we used the binomial distribution to directly calculate that $B_n g$ is rising when g is an increasing function. In general, we can demonstrate that while $g(x)$ is increasing (or decreasing), the derivative $(B_n g)'(x)$ of $B_n g(x)$ is positive (or negative), and as a result, $B_n g$ is likewise increasing (or reducing). We can actually offer the following [12]. We shall, presented some important properties of the derivative of the Bernstein polynomial .

Theorem 3.1. The derivative of the Bernstein polynomial $B_n g$ can be expressed as

$$(B_n g)'(x) = E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)}\, D(n,g)(x)\right], \qquad (3.1)$$

Where $$D(n,g)(x) = \frac{g\left(\frac{A_{n,x}}{n}\right) - g(x)}{\frac{A_{n,x}}{n} - x}.$$

Proof . Deriving (1.1) with respect to x we obtain

$$(B_n g)'(x) = \sum_{i=0}^{n} \binom{n}{i}\; g\left(\frac{i}{n}\right) x^{i-1}(1-x)^{n-i} - qw\backslash AW \sum_{i=0}^{n} \binom{n}{i} (n-i)\; g\left(\frac{i}{n}\right) x^{i}(1-x)^{n-i-1}$$

$$= \frac{1}{x} E A_{n,x} g\left(\frac{A_{n,x}}{n}\right) - \frac{1}{1-x} E\left(n - A_{n,x}\right) g\left(\frac{A_{n,x}}{n}\right)$$

$$= \frac{1}{x(1-x)} E\left[\left(A_{n,x} - nx\right) g\left(\frac{A_{n,x}}{n}\right)\right]$$

$$= \frac{1}{x(1-x)} E\left[\left(A_{n,x} - nx\right)\left(g\left(\frac{A_{n,x}}{n}\right) - g(x) + g(x)\right)\right]$$

$$= \frac{1}{x(1-x)} E\left[\left(A_{n,x} - nx\right)\left(g\left(\frac{A_{n,x}}{n}\right) - g(x)\right)\right]$$

$$= E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)} \left(\frac{g\left(\frac{A_{n,x}}{n}\right) - g(x)}{\frac{A_{n,x}}{n} - x}\right)\right]$$

$$= E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)}\; D(n,g)(x)\right],$$

as claimed.

Theorem 3.2. Assume $\sup_{x\in[0,1]} g(x) - M < \infty$. Then for any x such that $g'(x)$ exists we have

$$\lim_{n\to}(B_n g)'(x) = g'(x).$$

Proof. Define $A(n,g)(x) = D(n,g)(x) - g'(x)$. Then we can write

$$(B_n g)'(x) = E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)}\; D(n,g)(x)\right]$$

$$= g'(x) + E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)}\; A(n,g)(x)\right], \qquad (3.2)$$

Since $E\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)} = 1$.

Now, since $g'(x)$ exists, given $\epsilon > 0$, there exists $\delta > 0$ such that $\left\|\frac{A_{n,x}}{n} - x\right\| < 0\delta$
Then $|A(n,g)(x)| < \epsilon$. We prove that the absolute value of the second summand in (3.2) goes to zero by splitting and bounding it by

$$E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)} \ |A(n,g)(x)| 1_{\left\{\left|\frac{A_{n,x}}{n} - x\right| < \delta\right\}}\right]$$
$$+E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)} \ |A(n,g)(x)| \ 1_{\left\{\left|\frac{A_{n,x}}{n} - x\right| \geq \delta\right\}}\right], \qquad (3.3)$$

where 1_A represents the set A's indicator function. In (3.3), the first summand can be bounded by

$$E\left[\frac{\left(A_{n,x} - nx\right)^2}{nx(1-x)} \ 1_{\left\{\left|\frac{A_{n,x}}{n} - x\right| < \delta\right\}}\right] \leq \epsilon, \qquad (3.4)$$

Where the second summand in (3.3) can be bound by

$$\left(\frac{2M}{\delta} + g'(x)\right)\frac{n}{4}E\left[\left(\frac{A_{n,x}}{n} - x\right)^2 \ 1_{\left\{\left|\frac{A_{n,x}}{n} - x\right| \geq \delta\right\}}\right]$$
$$\leq \left(\frac{2M}{\delta} + g'(x)\right)\frac{n}{4}P\left(\left|\frac{A_{n,x}}{n} - x\right| \geq \delta\right), \qquad (3.5)$$

In the final inequality, the square of the distances between points $\frac{A_{n,x}}{n}$ and x in the interval [0,1] is smaller than 1.

$$\left(\frac{2M}{\delta} + g'(x)\right)\frac{n}{2}e^{-2n\delta^2},$$

Which goes to zero as $n \to \infty$, completing the proof.

Conclusion:

The summary of this research is the approximation of unbounded by using the Bernstein polynomial. in addition to the approximation of the derivative of unbounded functions by using the derivative of the Bernstein.

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