Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 4, No. 3, 2026, Pages 37-42
DOI: 10.47831/mjpas.v4i3.401

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage:
https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

The equivalent theorems for best approximation in weighted spaces

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Article Info.

Article history:

Received
26 March 2025

Accepted
13 May 2025

Publishing
30 September 2026

Abstract

The modulus of smoothness is used to demonstrate the direct and inverse trigonometric polynomials estimation theorem in weighted space with the degree of best approximation of unbounded functions. Also, we approximate of derivative function by derivative of trigonometric polynomials in weighted spaces. Moreover, an unbounded function is better the degree of best approximation for derivative functions in same space.

The official journal published by the College of Education at Mustansiriyah University

Keywords: Direct approximation theorem , inverse approximation theorem , best approximation , generalized modulus of smoothness, trigonometric polynomials

1. Introduction

The approximation theory, a branch of function analysis that is fundamental, is drawn to approximation functions by means of easier-to-calculate functions. Similar to other areas of mathematics, such as differential equations, finite elements, numerical analysis, and perpendicular polynomials, this theory has further implications. Applications include economic pressures, computer science, communication signals, and outline acknowledgment. The well-known works of Jackson (1911) and Bernstein (1912) established links between the variance and difference characteristics of the function being estimated and the value of the mistake of its estimate by certain techniques. The first theorems of direct and inverse approximation were discovered. Consequently, alike researches were conducted by several researchers for several functional classes and for various approximating aggregates, and their findings establish the classics of recent approximation principle. Furthermore, the exact findings (in particular, in the sense of un improvable constants) deserve special attention. A fairly whole explanation of the findings on finding direct and inverse estimation theorem is limited in the monographs [1, 2, 7, 10], etc. In 1972 Prytula [3] developed a direct approximation theorem for Besicovitch nearly intervallic equations of order 2 (B2-a.p. functions) in terms. The greatest estimates of equations and their modulus of continuity. G. V. Radzievskii investigated the direct and inverse theorem [4, 5], employing the concept of a K-functional rather than the module of continuity. Bonis and Mastroianni investigated the global expanding weights estimate using trigonometric polynomials in the periodic situation in 2003, as well as other related problems [6]. In 2006 Finta [8] analyzes direct approximation for discrete kind operators in terms of a modified K-functional. He showed direct theorem for Szász-Mirakjan, Lupa, and Baskakov kind operators.In 2008 Akgün and Daniyal [9] investigated approximation difficulties in Smirnov-Orlicz spaces using the fractional modulus of smoothness. In addition, to proving the direct and inverse theorems in these spaces and deriving a constructive description of the Lipschitz classes of function definite by the modulus of smoothness. In 2017, Rao [13] verified generalized Baskakov operators and studied their degree of approximation via moduli of continuity, order of estimate for derivatives, and established a direct theorem using the K-functional and Ditzian-Totik moduli of smoothness. Auad and Khrajan [14] computed the rate for the greatest trigonometric approximation of unbounded equations of one variable in the weighted space Lp,α $[-\pi, \pi]$ and analyzed the degree of best trigonometric approximation by modulus of smoothness in $\boldsymbol{X}p,\alpha$ $[-\pi, \pi]$.In 2021, SERDYUK [16] used Besicovitch-Stepanets spaces (BS^p) to derive the direct and inverse estimate theorem. In this work we provide direct and inverse approximation theorem concerning generalized moduli of smoothness and the optimal estimates of equations. The research of constant accuracy in Jackson-kind inequalities receives special emphasis. The reader is advised to consult the following references [11,12,19,21,22].

2-Prelinimaries

In this section ,we will mention some important basics related to the subject.

For $g \in L_{\theta,p}(X)$ with norm in weighted space as

$\|g\|_{\theta,p} = \left(\int_X |g(a)\theta(a)|^p\right)^{\frac{1}{p}} < \infty$, where θ is weighted function such that (1)

$\theta\colon [-\pi,\pi] \to R$ and its positive function.

The moduli of smoothness of the function $g \in L_{\theta,p}(X)$ is definite by:

$$\Omega_\phi(g,\phi)_{\theta,p} = \left\|\Delta_\phi^{\mathrm{k}}(g,.)\right\|_{\theta,p} \quad ,\phi > 0 \tag{2}$$

The grade of best approximation of $g \in L_{\theta,p}(X)$ by

$$E_m(g)_{\theta,p} = \inf \|g - T_{m-1}\|_{\theta,p} \tag{3}$$

the space of all trigonometric polynomials which the degree m where T sup space of $L_{\theta,p}(X)$.

There are numerous finding on the approximation of function that belong to $L_{\theta,p}(X)$ spaces, $1 \leqslant p < \infty$

Partical arty, the classical Jackson theorem (direct theorem) [17].

$$E_m(g)_{\theta,p} \leqslant C\, \Omega_m\left(g,\frac{\pi}{m}\right)_{\theta,p} \quad , \quad m = 1,2,\ldots \tag{4}$$

and its feeble inverse

$$\Omega_m\left(g,\frac{\pi}{m}\right)_{\theta,p} \leq \frac{C}{m^{2j}} \sum_{n=0}^{m} (n+1)^{2j-1} E_m(g)_{\theta,p} \quad ,m = 1,2,\ldots \tag{5}$$

The averge modulus of smoothness denoted by $\tau(g,\phi)_{\theta,p}$, $1 \leq p < \infty$

is $\tau_m(g,\phi)_{\theta,p} = \|\Omega_m(f,\phi)\|_{\theta,p}$, we can show that

$$\tau_m(g,\phi)_{\theta,p} \cong \Omega_m(f,\phi)_{\theta,p} \tag{6}$$

in weighted space

3. Auxiliary Lemmas

In this section, we will discuss several of the lemmas that will be required to prove the main outcome theorem.

Lemma 3.1[15] Let $g \in L_p(X)$, $X \in [-\pi,\pi]$ and $1 \leq p < \infty$ .Then

(i) $\Omega_m(g,\phi)_p \geqslant 0$

(ii) $\Omega_m(g,\phi)_p \to 0$ as $\phi \to 0$,

$\phi = \frac{\pi}{m}$, $\frac{\pi}{m} \to 0$, $as \;\; m \to \infty$

(iii) $\Omega_m(g,\lambda\phi)_p \leqslant \lambda^n \Omega_m(g,\phi)_p$ $,\lambda > 0$

(iv) $\Omega_m(g+z,\phi)_p \leqslant \Omega_m(g,\phi)_p + \Omega_m(z,\phi)_p$ and

$\tau_n(g+z,\phi)_p \leqslant \tau_n(g,\phi)_p + \tau_n(z,\phi)_p$

(v) if $\phi_1 \geqslant \phi_2$ $,\phi_1,\phi_2 > 0$ then $\Omega_m(g,\phi_1)_p \geqslant \Omega_m(g,\phi_2)_p$ and

$\tau_n(g,\phi_1)_p \geqslant \tau_n(g,\phi_2)_p$
(vi) $\tau_n(g,\phi)_p \leqslant \phi\tau_{n-1}\left(\dot{g};\frac{n}{n-1}\phi\right)_p$
(vii) $\tau_n(g,\phi)_p \leq f(n)\ \phi^n\left\|g^{(n)}\right\|_p, if\ g^{(n)} \in L_{\theta,p}(X)$, $f(n)$ denotes a constant depending only on n
(viii) let $\mu,\phi > 0$, $1 \leqslant p < \infty$. Then
$\tau_n(g,\mu\phi)_p \leqslant f[\mu]^{n+1/\tilde{p}}\ \tau_n\ (g,\phi)_\rho$

where $\tilde{p} = \min(1,p)$, $[\mu] = \min\{\alpha \geqslant \mu\text{: } \alpha\ isan\ integer\}$ and the constant f depends on n and ϕ only
Lemma 3.2[15] Suppose that $\mathrm{P} = \{p_n\}_{n=-\infty}^{\infty}$ and $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}$ are non-negative number sequences in which
$1 < \mathrm{P} \leqslant \infty$, $n \in \mathbb{N}$. There fore, for every function $g \in L_{\theta,p}(X)$
$\|g\|_{\theta,p} \leqslant \mathrm{C(P)}\|g^{(p_n)}\|_{\theta,p}$, C(P) is positive constant

4. The Main Results

In this section, we present three formulas which of through approximate unbounded functions in weighted space via moduli of smoothness.

Theorem 4.1 let $\psi = \{\psi_n\}_{n=-\infty}^{\infty}$ be a random series of integers in such a way that $\psi_n \neq 0$ and $\lim_{|n|\to\infty}|\psi_n| = 0$. If for a function $g \in L_{\theta,p}(X)$ there is a derivative $g^{(\psi)} \in L_{\theta,p}(X)$, consequently the following inequality is true:
$E_m(g)_{\theta,p} \leqslant \mathrm{C}\ E_m\left(g^{\psi}\right)_{\theta,p}$, where $C = \max_{|n|\geqslant m}|\psi_n|$

Proof: According to (2.3) , we possess

$$E_m(g)_{\theta,p} = \inf\left\{n > 0\text{: } \int_{-\pi}^{\pi}\alpha_n\left(\frac{|\psi_n g^{\psi}(t)|}{n}\ d(t)\right)^{p_n} \leqslant 1\right\}$$
$$\leq\ \inf\left\{n > 0\text{: } \int_{-\pi}^{\pi}\alpha_n\left(\frac{C|g^{\psi}(t)|}{n}\ d(t)\right)^{p_n} \leqslant 1\right\}$$
$$\leq C\ E_m(g^{\psi})_{\theta,p}$$

Theorem 4.2 Suppose that $P = \{p_n\}_{n=-\infty}^{\infty}$ and $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}$ are non-negative number sequence in which $1 < p_n \leqslant \mathrm{T}$, $n \in \mathbb{N}$ and the function $g \in L_{\theta,p}(X)$. After that for any number $\beta > 0$ and $m \in \mathbb{N}$. the following inequality is true:
$E_m(g)_{\theta,p} \leqslant C(\beta)\Omega_m(g,m^{-1})_{\theta,\rho}$
Where $C = C(\beta)$ is a constant independent of g and m

proof: Let $\{T_m(a)\}_{m=1}^{\infty}$ represent aseries of kernels (where T_m (a) is a polynomial of trigonometry with an order of m or loss). Satisfying to every one $m = 1,2,\ldots$ the condition:

$$\int_{-\pi}^{\pi}T_m(a)\ da = 1 \tag{7}$$

$$\int_{-\pi}^{\pi}|a|^j|T_m(a)|da \leqslant C(j)(m+1)^{-j} \quad ,j = 0,1,2,\ldots \tag{8}$$

we can specifically use the well- known Jakson kernels of sufficiently high order in the function of such kernels, that is
$T_m(a) = c_q\left(\frac{\sin pa/2}{\sin a/2}\right)^{2n_0}$
where n_0 is an independent integer of $m, 2n_0 \geqslant j+2$, the inequality is used to get the positive integer p, $m/(2n_0) < p \leqslant m/(2n_0)+1$, and the constant c_q is selected in accordance with the normalizing criterion (7) , it was demonstrated that for any kernel $\{\ T_m(a)\}$ sequence meeting requirements $(7)-(8)$, the next estimatation is accurate:

$$\int_{-\pi}^{\pi}(|a|+m^{-1})^{j}|T_m(a)|da \leqslant C(j)m^{-j},\ (j,m=1,2,\dots) \tag{9}$$

first, let's look at the instance of $\beta \in \mathbb{N}$.set

$$\phi_{m-1}(b)=(-1)^{\beta+1}\int_{-\pi}^{\pi}T_{m-1}(a)\sum_{k=1}^{\beta}(-1)^{k}\binom{\beta}{k}g(b-\mathrm{k}\,\mathrm{a})da$$

Clearly ϕ_{m-1} (b) is a trigonometric polynomial with an order of less than m. Additionally, Consider (7)
we have

$$g(b)-\phi_{m-1}(b)=(-1)^{\beta}\int_{-\pi}^{\pi}T_{m-1}(a)\sum_{k=0}^{\beta}(-1)^{k}\binom{\beta}{k}g(b-ka)da=(-1)^{\beta}\int_{-\pi}^{\pi}T_{m-1}(a)\Delta_a^{\beta}g(b)da$$

Therefore, from lemma (3.1) and the definition of the set F, we get

$$E_m(g)_{\theta,p}\leqslant\|g-\phi_{m-1}\|_{\theta,p}=\left\|(-1)^{\beta}\int_{-\pi}^{\pi}T_{m-1}(a)\Delta_a^{\beta}g\,da\right\|_{\theta,p}$$

$$=\sup\left[\sum_{n\in\mathbb{N}}\alpha_n\,\mu_n\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}T_{m-1}(a)\ \Delta_a^{\beta}\,g(b)\ da\right)e^{-nb}db\right|:\mu\in\mathrm{F}\right]$$

Currently, using the Fubini formula and estimation (3.1) once more, we discover

$$E_m(g)_{\theta,p}\leqslant\int_{-\pi}^{\pi}|T_{m-1}(a)|\ \sup\left[\sum_{n\in\mathbb{N}}\alpha_n\,\mu_n\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_a^{\beta}\,g(b)e^{-nb}db\right|:\mu\in\mathrm{F}\right]da$$

$$\leqslant 2\int_{-\pi}^{\pi}|T_{m-1}(a)|\ \left\|\Delta_a^{\beta}\,g(b)\right\|_{\theta,p}da$$

$$\leqslant 2\int_{-\pi}^{\pi}|T_{m-1}(a)|\ \Omega_m(g,|a|)_{\theta,p}da \tag{10}$$

we estimation the integral on the right-hand side of relative (10). Configuring $\phi=m^{-1}$, we can see $\Omega_m(g;|a|)_{\theta,\mathrm{p}}\leqslant m^{\beta}(|a|+m^{-1})^{\beta}\Omega_m(g,m^{-1})_{\theta,p}$ .Using (9) and this inequality we obtain

$$\int_{-\pi}^{\pi}|T_{m-1}(a)|\,\Omega_m(g,|a|)_{\theta,p}da\leqslant m^{\beta}\Omega_m(g,m^{-1})_{\theta,p}\int_{-\pi}^{\pi}(|a|+m^{-1})^{\beta}|T_{m-1}(a)|da\leqslant C(\beta)\Omega_m(g,m^{-1})_{\theta,p}$$

consequently, the theorem is proven in the case of $\beta\in\mathbb{N}$

Theorem 4.3 If $g\in L_{\theta,p}(X)$, therefore for any $\beta>0$, and $m\in\mathbb{N}$, the following inequality is accurate :

$$\Omega_m\left(g,\frac{\pi}{m}\right)_{\theta,\mathrm{p}}\leqslant\left(\frac{\pi}{m}\right)^{\beta}\sum_{r=1}^{m}\left(r^{\beta}-(r-1)^{\beta}\right)E_r(g)_{\theta,p} \tag{11}$$

Proof: Let's adapting if to the unique characteristics of the spaces $L_{\theta,p}(X)$.
Let $g\in L_{\theta,p}(X)$ $,k\in\mathbb{N}^0$ and $g_{kt}(b)=g(b-kt)$, where $k=0,1,\dots$ and $t\in\mathbb{R}$. Then for any $n\in\mathbb{N}$, we possess $g_{kt}(n)=\mathrm{g}(n)e^{-nkt}$.

$$\{\Delta_t^{\beta}g\}(n)=\{\sum_{k=0}^{\infty}(-1)^{k}\binom{\beta}{k}g_{kt}\}(n)=\mathrm{g}(n)\sum_{k=0}^{\infty}(-1)^{k}\binom{\beta}{k}e^{-nkt}=(1-e^{-nt})^{\beta}\mathrm{g}(n) \tag{12}$$

And

$$\sup\left\{\left|\Delta_t^{\beta}g(n)\right|\right\}=\left|\{\Delta_t^{\beta}g\}(n)\right|=|1-e^{-nt}|^{\beta}\,|\mathrm{g}(n)|=2^{\beta}\left|\sin\frac{nt}{2}\right|^{\beta}|\mathrm{g}(n)| \tag{13}$$

Since $g\in L_{\theta,p}(X)$,therefore for every $\varepsilon>0$ there be present a number $M_0=M_0(\epsilon)\in\mathbb{N},\ M_0>m$, such that for any $M>M_0$, we have $E_M(g)_{\theta,p}=\|g-G_{M-1}(g)\|_{\theta,p}<2^{-\beta}\epsilon$. Let's set $g_0=G_{M_0}(g)$. Then

Considering (13), we observe that
By characteristic of symmetric differences

$$\left\|\Delta_t^{\beta}g\right\|_{\theta,p}\leqslant\left\|\Delta_t^{\beta}g_0\right\|_{\theta,p}+\left\|\Delta_t^{\beta}(g-g_0)\right\|_{\theta,p}\leqslant\left\|\Delta_t^{\theta}g_0\right\|_{\theta,p}+2^{\beta}E_{M_0+1}(g)_{\theta,p}<\left\|\Delta_t^{\beta}g_0\right\|_{\theta,p}+\varepsilon \tag{14}$$

Additionally, let $G_{m-1}=G_{m-1}(g_0)$ represent the fourier sum of g_0. Consequently by virtue of (13) of $|t|\leqslant\pi/\mathrm{m}$, we have

$$\left\|\Delta_t^{\beta}g_0\right\|_{\theta,p}=\left\|\Delta_t^{\beta}(g_0-G_{m-1})+\Delta_t^{\beta}\,G_{m-1}\right\|_{\theta,p}\leq\left\|2^{\beta}(g_0-G_{m-1})+\sum_{|n|\leq m-1}|nt|^{\beta}|\mathrm{g}(n)|e^{n}\right\|_{\theta,p}$$

$$\leq \left\| 2^{\beta}(g_0 - G_{m-1}) + \left(\frac{\pi}{m}\right)^{\beta} \sum_{|n|\leq m-1} |n|^{\beta}|g(n)|e^{n} \right\|_{\theta,p}$$

$$= \left\| 2^{\beta} \sum_{r=m}^{M_0} D_r + \left(\frac{\pi}{m}\right)^{\beta} \sum_{r=1}^{m-1} r^{\beta} D_r \right\|_{\theta,p} \qquad (15)$$

Where $D_r(b) = D_r(g,b) = |g(r)|e^{rb} + |g(-r)|e^{-rb}$, $r = 1,2,\ldots$

Using the following claim, which has direct proof.
Let $\{f_r\}_{r=1}^{\infty}$ and $\{t_r\}_{r=1}^{\infty}$ be any random number sequence. Then, all natural l have the following equality, W and
$l \leqslant W < M$:

$$\sum_{r=l}^{W} t_r f_r = t_l \sum_{r=l}^{M} f_r + \sum_{r=l+1}^{W} (t_r - t_{r-1}) \sum_{s=r}^{M} f_s - t_w \sum_{r=w+1}^{M} f_r \qquad (16)$$

putting $t_r = r^{\beta}, f_r = D_r(b)$, $l = 1$, $W = m-1$ and $M = M_0$ in (16) we obtain

$$\sum_{r=1}^{m-1} r^{\beta} D_r(b) = \sum_{r=1}^{M_0} D_r(b) + \sum_{r=2}^{m-1}\left(r^{\beta} - (r-1)^{\beta}\right) \sum_{s=r}^{M_0} D_s(b) - (m-1)^{\beta} \sum_{r=m}^{M_0} D_r(b)$$

Therefore,

$$\left\| 2^{\beta} \sum_{r=\mathrm{m}}^{M_0} D_r + \left(\frac{\pi}{m}\right)^{\beta} \sum_{r=1}^{\mathrm{m}-1} r^{\beta} D_r \right\|_{\theta,p} \leqslant \left(\frac{\pi}{m}\right)^{\beta} \left\| m^{\beta} \sum_{r=\mathrm{m}}^{M_0} D_r + \sum_{r=1}^{\mathrm{m}-1} \left(r^{\beta} - (r-1)^{\beta}\right) \sum_{s=r}^{M_0} D_s - (m-1)^{\beta} \sum_{r=m}^{M_0} D_r \right\|_{\theta,p}$$

$$\leqslant \left(\frac{\pi}{m}\right)^{\beta} \left\|\sum_{r=1}^{m} \left(r^{\beta} - (r-1)^{\beta}\right) \sum_{s=r}^{M_0} D_s \right\|_{\theta,p} \leqslant \left(\frac{\pi}{m}\right)^{\beta} \sum_{r=1}^{m} \left(r^{\beta} - (r-1)^{\beta}\right) E_r(g_0)_{\theta,p} \qquad (17)$$

By combining relations (14), (15) and (17) and considering the description of the function g_0, as we can see for $|t| \leqslant \pi/\mathrm{m}$. This inequality is true:

$$\left\|\Delta_t^{\beta} g\right\|_{\theta,p} \leqslant \left(\frac{\pi}{m}\right)^{\beta} \sum_{r=1}^{m} \left(r^{\beta} - (r-1)^{\beta}\right) E_r(g)_{\theta,p} + \varepsilon$$

5.CONCLUSION

It was concluded that the degree of best approximation for the unbounded function in the weighted space is better for approximating functions than the degree of best approximation for the derivative of the function in the same space. Moreover, the direct theorem for approximating the unbounded function under the conditions of the module of smoothness was proven, and finally the inverse theorem for the unbounded function in the weighted space was proven.

Acknowledgement
I would like to thank Professor Dr. Alaa Adnan Awad for his help in completing this research and producing it in the best possible way. He also enriched the research with the necessary amendments.

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