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RESEARCH ARTICLE - MATHEMATICS

Advanced Fixed point Analysis in Partial Metric Type Spaces under Fuzzy soft set Theoretic Constraints

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Abstract

In this paper ,the concept of fuzzy soft $\mathbb{b}$ -partial Hausdorff Metric are considered , the fixed point theorems of Banach contraction principle in fractional form are generalized to complete fuzzy soft $\mathbb{b}$ -partial metric space, and prove that a single valued $\mathfrak{f}_{ss}$ -- mapping ,multi valued $\mathfrak{f}_{ss}$- mapping have a $\mathfrak{f}_{ss}$- fixed point in this space.

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1. Introduction

Functional analysis plays a fundamental and important role in the development of applied sciences, as its modern applications are used to analyze complex problems. Therefore, it is considered a fruitful field of study for many researchers; see [1-6]. The fuzzy theory presented by Zadeh in 1965 [7], attracted the attention of researchers in the field of artificial intelligence ,as he showed through his theory that an element can belong to a set partially through the membership fuction ,and this is considered his development of the classical concept. After that, many researchers worked on fuzzy theory with different research trends, see[8-12]. In 1999 , Molodtsov challenged traditional theories by introducing a soft set, which is flexible in dealing with time-varying data to obtain accurate results [13].In 1994 ,Matthews [14]created partial metric spaces and introduced a relation in this space through which we can generate classical spaces .The importance of the fixed point is not limited to proving the existence of solutions to differential equations without solving them ,but rather its importance extends to being an effective tool for studying the stability of some dynamic systems, and it has various applications in physics, computers ,and functional analysis. In 2019, Hashim and Bakry, [15] circulated the common fixed point theories under the influence of Ciric contractions in this spaces of Partial b-Metric with applied examples in these spaces. In 2023, Kumari et al., [16] circulated the applied results of the fixed point theory for Kannan contraction in the strong partial b- metric spaces, expanding the principle of Banach contraction. In 2024. Massit and Rossafi [17] proved the existence of single fixed point for the integral equations in the b- metric space with algebraic values C^*. This research is considered abridge between the concepts of abstract algebra and the fixed point theory.In 2025,Rani et al., [18] presented KransnoselsKij repetitive method, supported by illustrative examples, to confirm the efficiency of that method in the convex G- metric space, where they proved the existence of single fixed points under the influence of expanded contraction in this space. Tarte and Borgaonkar in 2025[19] contribute to the development of the classic concept of fixed point theory by adjusting the contraction in the compact metric spaces, as the proved that self -derives have unique fixed point in that space. Fuzzy soft is one of the most powerful and modern mathematical tools ,combining fuzzy theory and soft sets, which gives it great flexibility in analysis ,as it combines the theoretical and applied aspect ,thus attracting thee interest of many researchers to work in this modern And flexible field see [20-28].

Now we will introduce the definition of fuzzy soft 𝕓 - Partial metric space, which is a generalization of the definition that appeared in [29] ,where in this study the fuzzy soft fixed point of the contractionary functions defined on this space is found.

2. Preliminaries

In this section we will provide some basic definitions about fuzzy and soft sets as well as some fuzzy soft metric spaces related to 𝕓 -partial metric space($\mathcal{bpms}$).

Definition 2.1: [7] Let $\mu_{\tilde{X}}(u)$ Be the degree membership of $\mathcal{u}$ in $\tilde{X}$ in to I = [0, 1], U be a universal set and . Then the fuzzy set $\tilde{X}$ can be defined by ordered pair $\tilde{X}$={($(\mathcal{u}, \mu_{\tilde{X}}(u))$: $\mathcal{u} \in U, \mu_{\tilde{X}}(u) \in I$)}.

Definition 2.2: [13] The soft set of U is the ordered pair (G, A)can be defined as $G_A = \{(e, G_A(e): G_A(e) \in P(U), e \in E\}$ **,**where G:A→ $P(U)$ be mapping , $A \subseteq E$(collection of parameters) and $P(U)$ be a power set of U.

Definition 2.3: [21] The fuzzy soft is the ordered pair (G, A)can be defined as the collection$\{e \in A, G_A(e) \in I^U\}$

where G:A→ I^U be mapping, and denoted by $\mathfrak{f}_{ss}$.

Definition 2.4: [22]

1- If $\mu G(e) = 1$ for all $e \in A$ then $\mathfrak{f}_{ss} - (G, A)$ is said to be absolute $-\mathfrak{f}_{ss}$, symbolized by $\widetilde{C_A}$.

2- If $\mu G(e) = \tilde{0}$ for all $e \in A$ then $\mathrm{F}_{ss} - (G, A)$ is said to be null$-\mathfrak{f}_{ss}$, symbolized by $\tilde{\emptyset}$.

Definitions 2.5: [21] Let (G_1, A) and (G_2, B) be two $\mathfrak{f}_{ss}$ for common universal set U ,then:

1- $(G_1, A) \subseteq (G_2, B)$ if for all $e \in A$ we have $A \subseteq B$, and $G_1(e) \subseteq G_2(e)$, $\mu G_1(e) \leq \mu G_2(e_2)$.

2- (G_1, A) =(G_2, B)if $(G_1, A) \subseteq (G_2, B)and(G_2, B) \subseteq (G_1, A)$.

Definition 2.6: [23] If$\mu G(e) = \begin{cases} \rho \text{ if } \upsilon = \upsilon_0 \text{ and } e = e_0 \in A \\ 0 \ \ if \ u \in U - \{\upsilon_0\} \text{ or}, e \in A - \{e_0\} \text{ where } \rho \in (0,1] \end{cases}$

when $\upsilon \in U$ and e$\in A$, then $\mathfrak{f}_{ss} - (G, A)$ is said to be $\mathfrak{f}_{ss}$ $-$point and symbolized by $\upsilon_{\mu G(e)}$.

Definition 2.7: [24] 1. The $\mathfrak{f}_{ss}$ $-$point $\mathcal{u}_{\mu G(e)}$ is said to be belong to (G_1, A)if for all $\in A$,$G(e) \subseteq G_1(e)$.

2. The $\mathfrak{f}_{ss}$ $-$point ${\upsilon^1}_{\mu G(e_1)}$ is said to be equal to $\mathfrak{f}_{ss}$ $-$point ${\upsilon^2}_{\mu G(e_2)}$ if $\mu G_1(e_1) = \mu G_2(e_2)$ and if $\upsilon^1 = \upsilon^2$.

Definition 2.8: [25] The $\mathrm{F}_{ss} - (G_3, A)$ can be defined as a intersection of $\mathfrak{f}_{ss} - (G_1, A)$ and (G_2, B)

Where $\mu G_{3\,(e)}(\upsilon)$=$\begin{cases} \mu G_{1\,(e)}(\upsilon), \text{if } e \in A - B, \upsilon \in U \\ \mu G_{2\,(e)}(\upsilon) \text{if } e \in B - A, \upsilon \in U \\ min[\mu G_{1\,(e)}(\upsilon), \mu G_{2\,(e)}(\upsilon)] \text{ , if } e \in A \cap B, \upsilon \in U, \text{ where } \upsilon \in U, \mathrm{B} \cup \mathrm{A} = \mathrm{C} \text{ and } \mathrm{e} \in C \end{cases}$

Definition 2.9: [26] Let $\mathfrak{f}_{ss}(U)$ be a non empty $\mathfrak{f}_{ss}$ of U then $\tilde{\mathfrak{D}} : \mathfrak{f}_{ss}(U) \times \mathfrak{f}_{ss}(U) \to R^+(A)$ is said to be fuzzy soft metric on $\mathfrak{f}_{ss}(U)$ if for all ${\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}, {\upsilon^3}_{\mu G(e_3)} \in \mathfrak{f}_{ss}(U)$, the following conditions are satisfy:

1-$\tilde{\mathfrak{D}}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong \tilde{\mathfrak{D}}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)})$.

2-$\widetilde{\mathfrak{D}}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \geqq 0$.

3- $\widetilde{\mathfrak{D}}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong 0 \leftrightarrow {u^1}_{\mu\mathfrak{S}(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$.

4-$\widetilde{\mathfrak{D}}\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}\right) \leqq \left[\widetilde{\mathfrak{D}}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) + \widetilde{\mathfrak{D}}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^3}_{\mu G(e_3)})\right]$,

and $(\mathrm{F}_{ss}(U), \widetilde{\mathfrak{D}})$ is fuzzy soft metric space and denoted by$\mathfrak{f}_{ss} - \mathcal{ms}$.

Definition 2.10: [27] Let $\widetilde{\mathfrak{D}}^b{:}\, \mathfrak{f}_{ss}(U) \times \mathfrak{f}_{ss}(U) \to R^+(A)$ be function where $\mathfrak{f}_{ss}(U)$ is non- empty$\mathfrak{f}_{ss}$ of U and S≥ 1, then $\widetilde{\mathfrak{D}}^b$ said to fuzzy soft b- metric on $\mathfrak{f}_{ss}(U)$ if for all ${\upsilon^1}_{\mu G(e_1)}$, ${\upsilon^2}_{\mu G(e_2)}$, ${\upsilon^3}_{\mu G(e_3)} \in \mathfrak{f}_{ss}(U)$, the following conditions are satisfy:

1- $\widetilde{\mathfrak{D}}^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) \cong 0$ if and only if${\upsilon^1}_{\mu G(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$.

2-$\widetilde{\mathfrak{D}}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \geqq 0$.

3- $\widetilde{\mathfrak{D}}^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) \cong \widetilde{\mathfrak{D}}^b\left({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)}\right)$

4- $\widetilde{\mathfrak{D}}^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}\right) \leqq S\left[\widetilde{\mathfrak{D}}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) + \widetilde{\mathfrak{D}}^b({\upsilon^2}_{\mu G(e_2)}, {\upsilon^3}_{\mu G(e_3)})\right]$,

and $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}^b\right)$ fuzzy soft b- metric space and denoted by $\mathfrak{f}_{ss} - \mathcal{bms}$.

Definition 2.11: [28] Let $\mathfrak{f}_{ss}(U)$ is non- empty $\mathfrak{f}_{ss}$ of U and S ≥ 1 be real number, then $\widetilde{\mathfrak{D}}_p : \mathfrak{f}_{ss}(U) \times \mathfrak{f}_{ss}(U) \to R^+(A)$ is called fuzzy soft partial on $\mathfrak{f}_{ss}(U)$ if the following conditions are hold:

1)$\widetilde{\mathfrak{D}}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) \cong \widetilde{\mathfrak{D}}_p\left({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)}\right)$.

2)${\upsilon^1}_{\mu G(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$ if and only if $\widetilde{\mathfrak{D}}_p\left({\upsilon^2}_{\mu G(e_2)}, {\upsilon^2}_{\mu G(e_2)}\right) \cong \mathfrak{D}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^1}_{\mu G(e_1)}\right) \cong \mathfrak{D}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right)$.

3)$\widetilde{\mathfrak{D}}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) \geqq \widetilde{\mathfrak{D}}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^1}_{\mu G(e_1)}\right)$.

4)$\widetilde{\mathfrak{D}}_p\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) \leqq$

$\left[\widetilde{\mathfrak{D}_p}\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}\right) + \widetilde{\mathfrak{D}}_p\left({\upsilon^3}_{\mu G(e_3)}, {\upsilon^2}_{\mu G(e_2)}\right) - \widetilde{\mathfrak{D}}_p\left({\upsilon^3}_{\mu G(e_3)}, {\upsilon^3}_{\mu G(e_3)}\right)\right]$, for all ${\upsilon^1}_{\mu G(e_1)}$, ${\upsilon^2}_{\mu G(e_2)}$, ${\upsilon^3}_{\mu G(e_3)} \in \mathfrak{f}_{ss}(U)$and $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p\right)$ is a fuzzy soft partial metric space and denoted by $\mathfrak{f}_{ss} - \mathcal{pms}$.

3. Generalization of Banach̀s principle contraction with fuzzy soft 𝕓-partial metric space

In this section ,some of theorems of Banach contractionary principle are generalized to 𝕓 -partial metric space($\mathcal{bpms}$) under the influence of single-valued and multi-valued $\mathfrak{f}_{ss} -$ mapping with a variety of contractionary condition.

Definition 3.1

A function $\mathcal{T}\Omega = (\mathcal{T}, \Omega){:}\, (\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b) \to (\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b)$ is said to be $\mathfrak{f}_{ss} -$ contraction mapping if there exist $\tilde{\lambda} \in R^+(A)$ with $\tilde{0} \leq \tilde{\lambda} < \tilde{1}$ such that for each fuzzy soft point ${\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)} \in \mathfrak{f}_{ss}(U)$ we have $\widetilde{\mathfrak{D}}_p^b(\mathcal{T}\Omega({\upsilon^2}_{\mu G(e_2)}), \mathcal{T}\Omega({\upsilon^2}_{\mu G(e_2)})) \leq \tilde{\lambda}\, \tilde{d}_p^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right)$.

Now, let's take the following example to illustrate the definition.

Let $\mathrm{F}_{ss}(U) = R^+(A)$ and$\widetilde{\mathfrak{D}}_p^b$: $\mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ defined by:

$\widetilde{\mathfrak{D}}_p^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right) = \left|{\upsilon^1}_{\mu G(e_1)} - {\upsilon^2}_{\mu G(e_2)}\right|^4 + \left(\max\{{\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\}\right)^4$, then $(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b)$ is $\mathfrak{f}_{ss} - \mathcal{pms}$. with S=3 and $\mathcal{T}\Omega$ is be $\mathfrak{f}_{ss} -$ contraction mapping where $\mathcal{T}\Omega({\upsilon^1}_{\mu G(e_1)}) = \frac{{\upsilon^1}_{\mu G(e_1)}}{2}$

Since $\widetilde{\mathfrak{D}}_p^b\left(\mathcal{T}\Omega({\upsilon^1}_{\mu G(e_1)}), \mathcal{T}\Omega({\upsilon^2}_{\mu G(e_2)})\right) = \frac{\tilde{1}}{16}\widetilde{\mathfrak{D}}_p^b\left({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}\right)$, $\tilde{\lambda} \in R^+(A)$

Definition3.2

Let $\left(\mathrm{F}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be $\mathrm{F}_{ss}-b\wp m s$,and $\mathcal{CB}(\mathfrak{f}_{ss}(U)$ the collection of all non empty

$\mathfrak{f}_{ss}$ −closed and bounded subset of $\mathfrak{f}_{ss}(U)$then for all $\mathcal{M},\mathcal{N} \in \mathcal{CB}(\mathfrak{f}_{ss}(U)$,define $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) = \max\{sub_{a_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{M}} \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{N}\big), sub_{b_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{N}} \widetilde{\mathfrak{D}}_p^b\big(b_{\mu G(\mathrm{e})}, \mathcal{M}\big)\}$ is called the $\mathfrak{f}_{ss}$ −Hausdorff b-partial metric where $\widetilde{\mathfrak{D}}_p^b(x_{\mu G(\mathrm{e})}, \mathcal{M}) = \inf\,\{\widetilde{\mathfrak{D}}_p^b(x_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})})\ a_{\mu G(\mathrm{e})} \in \mathcal{M}\}$ is the distance of $\mathfrak{f}_{ss}$ −point $x_{\mu G(\mathrm{e})}$ to $\mathcal{M}$.

Definition 3.3

Let $\mathfrak{f}_{ss}(U)$ be any non empty $\mathfrak{f}_{ss}$ an element $x_{\mu G(\mathrm{e})}$ in $\mathfrak{f}_{ss}(U)$ is said to $\mathfrak{f}_{ss}$ − fixed point of a multi valued $\mathfrak{f}_{ss}$ −mapping , $\mathcal{T}\Omega\colon \mathrm{F}_{ss}(U) \to 2^{x_{\mu G(\mathrm{e})}}$,if $x_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{T}\Omega(x_{\mu G(\mathrm{e})})$

Where $2^{x_{\mu G(\mathrm{e})}}$, denote the collection of all non empty sub set of $\mathfrak{f}_{ss}(U)$

Remark 3.4

let$\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be $b\wp m s$ and $\mathcal{A}$ be any nonempty -$\mathfrak{f}_{ss}$ in$\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ then $a_{\mu G(\mathrm{e})} \widetilde{\in} \bar{\mathcal{A}}$

if and only if , $\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{A}\big) = \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big)$ where $\bar{\mathcal{A}}$ closure F_{ss} of $\mathcal{A}$ with respect to$\widetilde{\mathfrak{D}}_p^b$.

Proof. Let $a_{\mu G(\mathrm{e})}$ be arbitrary $\bar{\mathcal{A}}$ to prove $\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{A}\big) = \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big)$

Since $\bar{\mathcal{A}} = \left\{u_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathfrak{f}_{ss}(U) \colon \widetilde{\mathfrak{D}}_p^b\big(u_{\mu G(\mathrm{e})}, \mathcal{A}\big) = \widetilde{\mathfrak{D}}_p^b\big(u_{\mu G(\mathrm{e})}, u_{\mu G(\mathrm{e})}\big)\right\}$ and let $a_{\mu G(\mathrm{e})} \widetilde{\in} \bar{\mathcal{A}}$ by definition of $\bar{\mathcal{A}}$

$\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{A}\big) = \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big)$ must be hold.

Now ,to prove the converse ,let $\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{A}\big) \cong \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big)$ and let $u_{\mu G(\mathrm{e})}$ be any$\mathfrak{f}_{ss}$ −point in $\mathfrak{f}_{ss}(U)$

Such that $\widetilde{\mathfrak{D}}_p^b\big(u_{\mu G(\mathrm{e})}, \mathcal{A}\big) \cong \widetilde{\mathfrak{D}}_p^b\big(u_{\mu G(\mathrm{e})}, u_{\mu G(\mathrm{e})}\big)$ that is mean the definition of $\bar{\mathcal{A}}$ is hold ,therefore $a_{\mu G(\mathrm{e})} \widetilde{\in} \bar{\mathcal{A}}$.

Lemma 3.5 Let $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be $\mathfrak{f}_{ss}-b\wp m s$ .Then for any $\mathcal{M},\mathcal{N},\mathcal{W}\ \widetilde{\in}\ \mathcal{CB}(\mathfrak{f}_{ss}(U)$ we have:

1-$\delta_p^b(\mathcal{M},\mathcal{M}) \cong \sup\{\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big), a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}\}$

2- $\delta_p^b(\mathcal{M},\mathcal{M}) \widetilde{\leq}\ \delta_p^b(\mathcal{M},\mathcal{N})$

3- $\delta_p^b(\mathcal{M},\mathcal{N}) \cong \tilde{0}$ implies that $\mathcal{M} \widetilde{\subseteq} \mathcal{N}$

4-$\delta_p^b(\mathcal{M},\mathcal{N}) \widetilde{\leq}\ S[\delta_p^b(\mathcal{M},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{N}) - inf_{w_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{W}} \widetilde{\mathfrak{D}}_p^b\big(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})}\big)]$

Proof.(1) from Remark 3.4**,** if $\mathcal{M} \widetilde{\in}\ \mathcal{CB}(\mathfrak{f}_{ss}(U)$,then for all $a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}$, we have

$\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big) \cong \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{A}\big)$ and $\bar{\mathcal{A}} \cong\ \mathcal{A}$.

So $\delta_p^b(\mathcal{M},\mathcal{M}) \cong \sup\left\{\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big), a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}\right\} =$

$\delta_p^b(\mathcal{M},\mathcal{M}) = \sup\{\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{M}\big), a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}\}$.

(2)Let $a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}$,since $\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big) \widetilde{\leq} \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}\big)$ for all $b_{\mu G(\mathrm{e})} \in \mathcal{N}$

So$\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big) \widetilde{\leq} \widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, \mathcal{N}\big) \widetilde{\leq} \delta_p^b(\mathcal{M},\mathcal{N})$. From (1), we have

$\delta_p^b(\mathcal{M},\mathcal{M}) \cong \sup\{\widetilde{\mathfrak{D}}_p^b\big(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}\big), a_{\mu G(\mathrm{e})} \widetilde{\in}\ \mathcal{M}\} \widetilde{\leq}\ \delta_p^b(\mathcal{M},\mathcal{N})$

(3)Let $\delta_p^b(\mathcal{M},\mathcal{N}) \cong \tilde{0}$ for all $a_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{M}$, that is$\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})},\mathcal{N}) \cong \tilde{0}$

For all $a_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{M}$,from (1) and (2) we have $\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}) \tilde{\leq} \delta_p^b(\mathcal{M},\mathcal{N}) = \tilde{0}$

for all $a_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{M}$, that is$\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})}) \cong \tilde{0}$, so $\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})},\mathcal{N}) \cong \tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, a_{\mu G(\mathrm{e})})$

For all $a_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{M}$by Remark 3.4**,** $a_{\mu G(\mathrm{e})} \in \bar{\mathcal{N}} \cong \mathcal{N}$,so $\mathcal{M} \tilde{\subseteq} \mathcal{N}$

(4) Let$a_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{M}, b_{\mu G(\mathrm{e})} \in \mathcal{N}$and $w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}$

$\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}) \tilde{\leq} S[\, \tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})}) + \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}) - \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})]$

so$\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})},\mathcal{N}) + S\tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})}) \tilde{\leq} S\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})}) + \delta_p^b(w_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})})]$

since $w_{\mu G(\mathrm{e})}$ is any $\mathfrak{f}_{ss}$ −elment of $\mathcal{W}$

so, $\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})},\mathcal{N}) + S\{inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})\} \tilde{\leq}$

$S[\tilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{N})]$

since $a_{\mu G(\mathrm{e})}$ is any $\mathfrak{f}_{ss}$ −elment of $\mathcal{M}$

so, $\delta_p^b(\mathcal{M},\mathcal{N}) \tilde{\leq} S[\delta_p^b(\mathcal{M},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{N}) - inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})]$

Lemma 3.6 Let $\mathcal{M},\mathcal{N},\mathcal{W} \tilde{\in} \mathcal{CB}(\mathfrak{f}_{ss}(U)$,then :

(1) $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \tilde{\leq} \mathcal{H}_p^b(\mathcal{N},\mathcal{M})$

(2) $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \tilde{\leq} \mathcal{H}_p^b(\mathcal{M},\mathcal{N})$

(3)) $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \tilde{\leq}$ S $[\mathcal{H}_p^b(\mathcal{M},\mathcal{W}) + \mathcal{H}_p^b(\mathcal{W},\mathcal{N}) - \inf\{\tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, \mathrm{w}_{\mu G(\mathrm{e})}) : w_{\mu G(\mathrm{e})} \in \mathcal{W}\}]$

Proof . By (2) of Lemma (3.5) Then $\mathcal{H}_p^b(\mathcal{M},\mathcal{M}) \cong \delta_p^b(\mathcal{M},\mathcal{M}) \tilde{\leq} \delta_p^b(\mathcal{M},\mathcal{N})$

$\tilde{\leq} \mathcal{H}_p^b(\mathcal{M},\mathcal{N})$. By (4) of Lemma (3.5.)

$\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \cong$ max{ $\delta_p^b(\mathcal{M},\mathcal{N}), \delta_p^b(\mathcal{N},\mathcal{M})\}$

$\tilde{\leq}$ max$\{S(\, \delta_p^b(\mathcal{M},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{N}) - inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})),$

S($\delta_p^b(\mathcal{N},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{M}) - inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})}))\}$

$\cong$ max$\{\boldsymbol{S}(\delta_p^b(\mathcal{M},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{N}) + \delta_p^b(\mathcal{N},\mathcal{W}) + \delta_p^b(\mathcal{W},\mathcal{M}))\} - S\{inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})\}$

$\tilde{\leq}$ max{S($\delta_p^b(\mathcal{M},\mathcal{W}), \delta_p^b(\mathcal{W},\mathcal{M})$)**}**+ max{S($\delta_p^b(\mathcal{W},\mathcal{N}), \delta_p^b(\mathcal{N},\mathcal{W})$)}

$-S\{inf_{w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}} \tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, w_{\mu G(\mathrm{e})})\}$

$\cong S\mathcal{H}_p^b(\mathcal{M},\mathcal{W}) + S\mathcal{H}_p^b(\mathcal{W},\mathcal{N}) - S\{\inf\{\tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, \mathrm{w}_{\mu G(\mathrm{e})}) : w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}\}\}$

$\cong$ S $[\mathcal{H}_p^b(\mathcal{M},\mathcal{W}) + \mathcal{H}_p^b(\mathcal{W},\mathcal{N}) - \inf\{\tilde{\mathfrak{D}}_p^b(w_{\mu G(\mathrm{e})}, \mathrm{w}_{\mu G(\mathrm{e})}) : w_{\mu G(\mathrm{e})} \tilde{\in} \mathcal{W}\}]$

Corollary 3.7 Let $\left(\mathfrak{f}_{ss}(U), \tilde{\mathfrak{D}}_p^b\right)$ be $\mathfrak{f}_{ss} - \mathscr{bpms}$,then for all $\mathcal{M},\mathcal{N} \tilde{\in} \mathcal{CB}(\mathfrak{f}_{ss}(U)$ if $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \cong \tilde{0}$ implies that $\mathcal{M} \cong \mathcal{N}$

Proof . Let $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) \cong \tilde{0}$**,** by definition of $\mathcal{H}_p^b$ **,** $\mathcal{H}_p^b(\mathcal{M},\mathcal{N}) = \delta_p^b(\mathcal{N},\mathcal{M}) \cong \tilde{0}$

By (3) of Lemma (3.5) ,we have $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{N} \cong \mathcal{M}$ Thus $\mathcal{M} \cong \mathcal{N}$.

Remark 3.8

The converse of corollary 3.7 is not necessary to be true ,to illustrate this, take the following example:

Example 3.9

Let where E={ e_1, e_2} , U=[0,1] and $\widetilde{\mathfrak{D}}_p^b: \mathfrak{f}_{ss}(U) \times \mathfrak{f}_{ss}(U) \rightarrow R(E)$ defined by:

$\widetilde{\mathfrak{D}}_p^b\left(u^1{}_{\mu G(e_1)}, {u^1}^2{}_{\mu G(e_2)}\right)$ =$(max\{v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\})^3$ then $(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b)$ is$\mathfrak{f}_{ss} - \mathscr{bpms}$ with S=2

$\mathcal{H}_p^b(\mathfrak{f}_{ss}(U), \mathfrak{f}_{ss}(U)) = \delta_p^b(\mathfrak{f}_{ss}(U), \mathfrak{f}_{ss}(U))= \tilde{1} \ncong \tilde{0}$.

Lemma 3.10

Let $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be $\mathrm{F}_{ss} - \mathscr{bpms}$, if $\mathcal{M}, \mathcal{N} \,\tilde{\in}\, BC(\mathfrak{f}_{ss}(U))$, then for any $a_{\mu G(\mathrm{e})} \,\tilde{\in}\, \mathcal{M}$, there $b_{\mu G(\mathrm{e})} \cong b_{\mu G(\mathrm{e})}(a_{\mu G(\mathrm{e})}) \,\tilde{\in}\, \mathcal{N}$ such that $\widetilde{\mathfrak{D}}_p^b\left(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}\right) \leq \mathfrak{h}\, \mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$here $\mathfrak{h} > 1$.

Proof. If $\mathcal{M} = \mathcal{N}$ then from (1) of Lemma 3.5, we get on

$\mathcal{H}_p^b(\mathcal{M}, \mathcal{N}) \cong \mathcal{H}_p^b(\mathcal{M}, \mathcal{M}) \cong \delta_p^b(\mathcal{M}, \mathcal{M}) \cong \sup_{u^1{}_{\mu G(e_1)} \in \mathcal{M}} \widetilde{\mathfrak{D}}_p^b(u^1{}_{\mu G(e_1)}, u^1{}_{\mu G(e_1)})$

Now , let $a_{\mu G(\mathrm{e})} \,\tilde{\in}\, \mathcal{M}$,since $\mathfrak{h} > 1$then we have

$\widetilde{\mathfrak{D}}_p^b\left(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}\right) \lessapprox \sup_{u^1{}_{\mu G(e_1)} \in \mathcal{M}} \widetilde{\mathfrak{D}}_p^b(u^1{}_{\mu G(e_1)}, u^1{}_{\mu G(e_1)}) \cong \mathcal{H}_p^b(\mathcal{M}, \mathcal{M}) \lessapprox \mathfrak{h}\mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$

Then $b_{\mu G(\mathrm{e})} \cong a_{\mu G(\mathrm{e})}$ satisfies $\widetilde{\mathfrak{D}}_p^b\left(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}\right) \leq \mathfrak{h}\, \mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$

If $\mathcal{M} \ncong \mathcal{N}$,let $a_{\mu G(\mathrm{e})} \,\tilde{\in}\, \mathcal{M}$ such that$\widetilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}) \gtrapprox \mathfrak{h}\mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$ for all $b_{\mu G(\mathrm{e})} \,\tilde{\in}\, \mathcal{N}$

Then $\inf\{\widetilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, b_{\mu G(\mathrm{e})}) : b_{\mu G(\mathrm{e})} \,\tilde{\in}\, \mathcal{N}\}] \gtrapprox \mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$. That is $\widetilde{\mathfrak{D}}_p^b(a_{\mu G(\mathrm{e})}, \mathcal{N}) \gtrapprox \mathfrak{h}\, \mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$

And $\mathcal{H}_p^b(\mathcal{M}, \mathcal{N}) \gtrapprox \delta_p^b(\mathcal{M}, \mathcal{N}) = \sup_{u^1{}_{\mu G(e_1)} \in \mathcal{M}} \widetilde{\mathfrak{D}}_p^b(u^1{}_{\mu G(e_1)}, \mathcal{N}) \gtrapprox \mathfrak{h}\mathcal{H}_p^b(\mathcal{M}, \mathcal{N})$

Since $\mathcal{M} \ncong \mathcal{N}$,from corollary (3.7) we get $\mathcal{H}_p^b(\mathcal{M}, \mathcal{N}) \ncong \tilde{0}$ and we have $\mathfrak{h} \leq 1$,this is contradiction.

Theorem 3.11

let $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be a complete F_{ss}- $\mathscr{bpms}$ and $\mathcal{T}\Omega$: $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right) \rightarrow \left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be $\mathfrak{f}_{ss}$ − mapping with coefficient S≥ 1 ,that is $\widetilde{\mathfrak{D}}_p^b(\mathcal{T}\Omega(v^1{}_{\mu G(e_1)}), \mathcal{T}(v^2{}_{\mu G(e_2)})) \lessapprox \tilde{\lambda}\, \widetilde{\mathfrak{D}}_p^b(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)})$ then

i) $\mathcal{T}\Omega$ has a $\mathfrak{f}_{ss}$ −fixed point $v_{\mu G(e)}$ in $\mathfrak{f}_{ss}(U)$

ii) If $v_{\mu G(e_0)}$ is an arbitrary point of $\mathfrak{f}_{ss}(U)$ and $\langle v^n{}_{\mu G(e_n)}\rangle$ is defined $v^{n+1}{}_{\mu G(e_{n+1})} \cong \mathcal{T}\Omega(v^n{}_{\mu G(e_n)})$, n= 0,1,2,…. ,then $\lim_{n\rightarrow\infty} v^n{}_{\mu G(e_n)} \cong v_{\mu G(e)}$ and $\widetilde{\mathfrak{D}}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)}) \lessapprox \frac{(S\,\tilde{\lambda})^n}{1-S\tilde{\lambda}} \widetilde{\mathfrak{D}}_p^b(v^1{}_{\mu G(e_1)}, v_{\mu G(e_0)})$

Proof. Let $\tilde{\lambda} \in [\tilde{0}, \tilde{1})$ be a $\mathfrak{f}_{ss}$ − contraction constant for $\mathcal{T}\Omega$ and $v_{\mu G(e_0)} \,\tilde{\in}\, \mathfrak{f}_{ss}(U)$

A sequence $\langle v^n{}_{\mu G(e_n)}\rangle$ is defined by $v^{n+1}{}_{\mu G(e_{n+1})} \cong \mathcal{T}\Omega(v^n{}_{\mu G(e_n)})$, n= 0,1,2,….

So ,by contraction property, one get

$$\widetilde{\mathfrak{D}}_p^b\left(v^{n+1}{}_{\mu G(e_{n+1})}, v^n{}_{\mu G(e_n)}\right) \lessapprox \tilde{\lambda}\, \widetilde{\mathfrak{D}}_p^b(v^n{}_{\mu G(e_n)}, v^{n-1}{}_{\mu G(e_{n-1})})$$

$$\lessapprox \tilde{\lambda}^2\, \widetilde{\mathfrak{D}}_p^b(v^{n-1}{}_{\mu G(e_{n-1})}, v^{n-2}{}_{\mu G(e_{n-2})})$$

$$\widetilde{\leq} \tilde{\lambda}^3 \, \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n-2}{}_{\mu G(e_{n-2})}, \upsilon^{n-3}{}_{\mu G(e_{n-3})}\right)$$

By repeating the same process we get:

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^{n}{}_{\mu G(e_n)}\right) \widetilde{\leq} \tilde{\lambda}^n \, \tilde{d}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$

Now to show that $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ be $\mathfrak{f}_{ss}$ $-$Cauchy sequence in in $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$

For all m$>$ n ,and by applying the triangular property of the $\mathcal{bpms}$,one get

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^n{}_{\mu G(e_n)}\right) \widetilde{\leq} S[\ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)$

$+\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^n{}_{\mu G(e_n)}\right) - \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)]$

$\widetilde{\leq} S[\ \tilde{d}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)\ +S^2(\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^{m-2}{}_{\mu G(e_{m-2})}\right)$

$+\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-2}{}_{\mu G(e_{m-2})}, \upsilon^m{}_{\mu G(e_m)}\right) - \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-2}{}_{\mu G(e_{m-2})}, \upsilon^{m-2}{}_{\mu G(e_{m-2})}\right)) - \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)]$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^n{}_{\mu G(e_n)}\right) \widetilde{\leq} S\ \ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)$

$+S^2\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^{m-2}{}_{\mu G(e_{m-2})}\right) +\ldots\ldots. +S^{m-n}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^n{}_{\mu G(e_n)}\right)$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^n{}_{\mu G(e_n)}\right) \widetilde{\leq} S\ \tilde{\lambda}^{m-1}\ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$

$+S^2\tilde{\lambda}^{m-2}\ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right) + \cdots +S^{m-n}\tilde{\lambda}^n\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^n{}_{\mu G(e_n)}\right)$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^n{}_{\mu G(e_n)}\right) \widetilde{\leq}\ \frac{S\,\tilde{\lambda}^n}{1-S\tilde{\lambda}}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$ this implies

$\frac{S\,\tilde{\lambda}^n}{1-S\tilde{\lambda}}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right) \to \tilde{0}$ as n $\to \infty$ that is mean $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ is a$\mathfrak{f}_{ss}$ $-$Cauchy sequence in in$\mathfrak{f}_{ss}(U)$ and since $\mathfrak{f}_{ss}(U)$ is complete , $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ to $\mathfrak{f}_{ss}$ $-$element $\upsilon_{\mu G(e)}$ in $\mathfrak{f}_{ss}(U)$.

Now ,to show that $\mathcal{T}\Omega\left(\upsilon_{\mu G(e)}\right) \cong \upsilon_{\mu G(e)}$

$\widetilde{\mathfrak{D}}_p^b\left(, \upsilon^n{}_{\mu G(e_n)}, \upsilon_{\mu G(e)}\right) \widetilde{\leq} S[\ \widetilde{\mathfrak{D}}_p^b\left(, \upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right) + \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon_{\mu G(e)}\right) -$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right)]$

Since $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon_{\mu G(e)}\right) \widetilde{\leq} \tilde{\lambda}^n\ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$

By repeat the same appreciation on $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon_{\mu G(e)}\right)$ then

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon_{\mu G(e)}\right) \widetilde{\leq} S\tilde{\lambda}^{n+1}\ \widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right) + S\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+2}{}_{\mu G(e_{n+2})}, \upsilon_{\mu G(e)}\right)$

By repeating the substitution we get

$\widetilde{\mathfrak{D}}_p^b\left(, \upsilon^n{}_{\mu G(e_n)}, \upsilon_{\mu G(e)}\right) \widetilde{\leq} S\ \tilde{\lambda}^n[1 + S\tilde{\lambda} + \left(S\tilde{\lambda}\right)^2 + \cdots]\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$

$\widetilde{\mathfrak{D}}_p^b\left(, \upsilon^n{}_{\mu G(e_n)}, \upsilon_{\mu G(e)}\right) \widetilde{\leq} \frac{(S\,\tilde{\lambda})^n}{1-S\tilde{\lambda}}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon_{\mu G(e_0)}\right)$

Theorem 3.12 let $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be a complete $\mathfrak{f}_{ss}$- $\mathcal{bpms}$ with coefficient S$\geq$ 1 and $\mathcal{T}\Omega$ be a single self valued $\mathfrak{f}_{ss}$ $-$ mapping defined on $F_{ss}(U)$ such that :

(1) $\widetilde{\mathfrak{D}}_p^b\left(\mathcal{T}\Omega\left(\upsilon^1{}_{\mu G(e_1)}\right), \mathcal{T}\Omega\left(\upsilon^2{}_{\mu G(e_2)}\right)\right) \widetilde{\leq}$

$$\frac{a\widetilde{\mathfrak{D}}_p^b\left(\upsilon^2{}_{\mu G(e_2)}, \mathcal{T}\Omega\left(\upsilon^1{}_{\mu G(e_1)}\right)\right)\left[1+\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)},\ \mathcal{T}\Omega\left(\upsilon^2{}_{\mu G(e_2)}\right)\right)\right]}{1+\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}\right)}$$

$+\ \mathfrak{b}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}\right)$ for each $\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)} \in \left(\tilde{0}, \tilde{1}\right)$ such that

$a + \mathfrak{b} < \tilde{1}$ and $S\left(\frac{\mathfrak{b}}{1-a}\right) < \tilde{1}$

(2) For some $\upsilon_{\mu G(e_0)} \in \mathfrak{f}_{ss}(U)$, the sequence $\langle \mathcal{T}\Omega(\upsilon_{\mu G(e_0)}\rangle$ has a subsequence $\langle \mathcal{T}\Omega^{n\mathcal{K}}\left(\upsilon_{\mu G(e_0)}\right)\rangle$ with $\upsilon^*{}_{\mu G(\mathrm{e})} = \lim_{n\to\infty} \mathcal{T}\Omega^{n\mathcal{K}}\left(\upsilon_{\mu G(e_0)}\right)$, then $\mathcal{T}\Omega$ has F_{ss} fixed point , that is $\upsilon^*{}_{\mu G(\mathrm{e})} = \mathcal{T}\Omega\left(\upsilon^*{}_{\mu G(\mathrm{e})}\right)$

Proof . Let $\upsilon_{\mu G(e_0)} \in \mathfrak{f}_{ss}(U)$ such that$\upsilon^1{}_{\mu G(e_1)} = \mathcal{T}\Omega\left(\upsilon_{\mu G(e_0)}\right)$ define the F_{ss} sequence $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ by $\upsilon^{n+1}{}_{\mu G(e_{n+1})} = \mathcal{T}\Omega(\upsilon^n{}_{\mu G(e_n)})$

$$\widetilde{\mathfrak{D}}_p^b\left(\mathcal{T}\Omega(\upsilon^{n-1}{}_{\mu G(e_{n-1})}), \mathcal{T}\Omega(\upsilon^n{}_{\mu G(e_n)})\right) \lesssim \frac{a\widetilde{\mathfrak{D}}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})})\left[1+\widetilde{\mathfrak{D}}_p^b(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)})\right]}{1+\widetilde{\mathfrak{D}}_p^b(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)})}$$

$$+\mathfrak{b}\widetilde{\mathfrak{D}}_p^b(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)})$$

$$(1-a)\widetilde{\mathfrak{D}}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right) \lesssim \mathfrak{b}\,\widetilde{\mathfrak{D}}_p^b(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)})$$

$$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right) \lesssim \frac{\mathfrak{b}}{(1-a)}\widetilde{\mathfrak{D}}_p^b(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)})$$

$$\lesssim \frac{\mathfrak{b}}{(1-a)}\left[\frac{\mathfrak{b}}{(1-a)}\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n-1}{}_{\mu G(e_{n-1})}, \upsilon^n{}_{\mu G(e_n)}\right)\right]$$

By repeating the same process we get:

$$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right) \lesssim \left(\frac{\mathfrak{b}}{(1-a)}\right)^{\boldsymbol{n}} \widetilde{\mathfrak{D}}_p^b\left(\upsilon_{\mu G(e_0)}, \upsilon^1{}_{\mu G(e_1)}\right)$$
(1)

To show that $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ be $\mathfrak{f}_{ss}$ −Cauchy sequence in in $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$

By using the trigonometric property in $bpms$ then we have

$$\widetilde{\mathfrak{D}}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}) \lesssim S[\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right) + \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}\right) - \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+1}{}_{\mu G(e_{n+1})}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right)]$$

By repeating the same process we have:

$$\widetilde{\mathfrak{D}}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}) \lesssim \mathrm{S}\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+1}{}_{\mu G(e_{n+1})}\right)$$
$$+ S^2\widetilde{\mathfrak{D}}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+2}{}_{\mu G(e_{n+2})}\right) + \boldsymbol{\ldots.}$$
$$+ S^m\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{n+m-1}{}_{\mu G(e_{n+m-1})}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}\right)$$
(2)

Substitute (1) in (2) then we have

$$\widetilde{\mathfrak{D}}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}) \lesssim \mathrm{S}\tilde{d}_p^b\left(\upsilon_{\mu G(e_0)}, \upsilon^1{}_{\mu G(e_1)}\right)\left(\frac{\mathfrak{b}}{(1-a)}\right)^{\boldsymbol{n}}\left[S + S^2\left(\frac{\mathfrak{b}}{(1-a)}\right) + \ldots\right.$$
$$\left. + S^m\left(\frac{\mathfrak{b}}{(1-a)}\right)^{\boldsymbol{m-1}}\right]$$

$$\widetilde{\mathfrak{D}}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}) \lesssim \frac{\boldsymbol{S}\left(\frac{\mathfrak{b}}{(1-a)}\right)^{\boldsymbol{n}}}{\boldsymbol{1}-\left(\frac{\mathfrak{b}}{(1-a)}\right)}\tilde{d}_p^b\left(\upsilon_{\mu G(e_0)}, \upsilon^1{}_{\mu G(e_1)}\right) \to \tilde{0} \text{ as } n \to \infty$$

Also, $\tilde{d}^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}\right) \le \widetilde{\mathfrak{D}}_p^b\left(\upsilon^n{}_{\mu G(e_n)}, \upsilon^{n+m}{}_{\mu G(e_{n+m})}\right) \to \tilde{0}$ as $n \to \infty$,that is mean $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is$\mathfrak{f}_{ss}$ −Cauchy sequence in in $\left(\mathfrak{f}_{ss}(U), \tilde{d}^b\right)$

Since in $\left(\mathfrak{f}_{ss}(U), \tilde{d}^b\right)$ is complete then $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ converges to $\upsilon_{\mu G(e_0)}$ in$\mathfrak{f}_{ss}(U)$.

Now, by using the trigonometric property in $bpms$ for each i we get

$\widetilde{\mathfrak{D}}_p^b \left(\upsilon_{\mu G(e_0 i)}, \mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)})\right) \mathrel{\tilde{\leqq}} S[\widetilde{\mathfrak{D}}_p^b \left(\upsilon_{\mu G(e_0 i)}, \mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)})\right) + \widetilde{\mathfrak{D}}_p^b \left(\mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)}), \mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)})\right) - \widetilde{\mathfrak{D}}_p^b \left(\mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)}), \mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)})\right)$

$\widetilde{\mathfrak{D}}_p^b \left(\upsilon_{\mu G(e_0 i)}, \mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)})\right) \mathrel{\tilde{\leqq}} S[\tilde{d}_p^b \left(\upsilon_{\mu G(e_0 i)}, \mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)})\right) \left(\mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)}), \mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)})\right) -$

$$\frac{a\widetilde{\mathfrak{D}}_p^b\left({\upsilon^n}_{\mu G(e_n)}, \mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)})\right)\left[1+\widetilde{\mathfrak{D}}_p^b\left({\upsilon^n}_{\mu G(e_n)}, \mathcal{T}\Omega({\upsilon^n}_{\mu G(e_n)})\right)\right]}{1+\widetilde{\mathfrak{D}}_p^b({\upsilon^n}_{\mu G(e_n)}, {\upsilon^n}_{\mu G(e_n)})}$$

$-\mathfrak{b}\widetilde{\mathfrak{D}}_p^b({\upsilon^n}_{\mu G(e_n)}, {\upsilon^n}_{\mu G(e_n)})] \to \tilde{0}$ as $n \to \infty$ that is

$\widetilde{\mathfrak{D}}_p^b \left(\upsilon_{\mu G(e_0 i)}, \mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)})\right) = \tilde{0}$,hence $\mathcal{T}\Omega(\upsilon_{\mu G(e_0 i)}) = \upsilon_{\mu G(e_0 i)}$ for each i.

Theorem 3.13 Let $\left(\mathfrak{f}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be a complete $\mathfrak{f}_{ss}$- $\mathcal{bpms}$ and $\mathcal{Z}\Omega, \mathcal{W}\Omega$ be a two set valued $\mathfrak{f}_{ss}$ − mapping defined on $\mathfrak{f}_{ss}(U)$ such that the contraction condition $\mathcal{H}_p^b\left(\mathcal{Z}\Omega(\mathrm{u}_{\mu G(\mathrm{e})}), \mathcal{W}\Omega(\mathrm{v}_{\mu G(\mathrm{e})})\right) \mathrel{\tilde{\leqq}} \mathfrak{h} L_p^b(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{v}_{\mu G(\mathrm{e})})$ where

$L_p^b(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{v}_{\mu G(\mathrm{e})}) = max\{\widetilde{\mathfrak{D}}_p^b(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{v}_{\mu G(\mathrm{e})}), \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{Z}\Omega(\mathrm{u}_{\mu G(\mathrm{e})})\right), \widetilde{\mathfrak{D}}_p^b\left(\mathrm{v}_{\mu G(\mathrm{e})}, \mathcal{W}\Omega(\mathrm{v}_{\mu G(\mathrm{e})})\right)\}$

And $\mathfrak{h} \mathrel{\tilde{\in}} (\tilde{0}, \tilde{1})$, then $\mathcal{Z}\Omega, \mathcal{W}\Omega$ have a $\mathfrak{f}_{ss}$-common fixed point

Proof. Let ${\mathrm{u}^0}_{\mu G(\mathrm{e}_0)} \mathrel{\tilde{\in}} \mathfrak{f}_{ss}(U), {\upsilon^1}_{\mu G(e_1)} \mathrel{\tilde{\in}} \mathcal{W}\Omega({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)})$,then we have two cases:

First:

If $L_p^b({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)}, {\upsilon^1}_{\mu G(e_1)}) = \tilde{0}$,then ${\mathrm{u}^0}_{\mu G(\mathrm{e}_0)} \mathrel{\tilde{\cong}} {\upsilon^1}_{\mu G(e_1)}$,so ${\mathrm{u}^0}_{\mu G(\mathrm{e}_0)} \mathrel{\tilde{\in}} \mathcal{W}\Omega({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)})$

Hence ${\mathrm{u}^0}_{\mu G(\mathrm{e}_0)}$ is a $\mathfrak{f}_{ss}$ − fixed point for $\mathcal{Z}\Omega, \mathcal{W}\Omega$.

Second:

If $L_p^b({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)}, {\upsilon^1}_{\mu G(e_1)}) \geq \tilde{0}$, then by lemma(3.10) with $\sigma > 0$ such that $\mathfrak{h} + \sigma < 1$

There is ${\upsilon^2}_{\mu G(e_2)} \mathrel{\tilde{\in}} \mathcal{Z}\Omega({\upsilon^1}_{\mu G(e_1)})$ such that

$\widetilde{\mathfrak{D}}_p^b({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)}) \mathrel{\tilde{\leqq}} \mathcal{H}_p^b\left(\mathcal{Z}\Omega({\upsilon^1}_{\mu G(e_1)}), \mathcal{W}\Omega({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)})\right) + \sigma L_p^b({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)}, {\upsilon^1}_{\mu G(e_1)})$

Also, there is ${\upsilon^3}_{\mu G(e_3)} \mathrel{\tilde{\in}} \mathcal{Z}\Omega({\upsilon^2}_{\mu G(e_2)})$ such that

$\widetilde{\mathfrak{D}}_p^b({\upsilon^3}_{\mu G(e_3)}, {\upsilon^2}_{\mu G(e_2)}) \mathrel{\tilde{\leqq}} \mathcal{H}_p^b\left(\mathcal{Z}\Omega({\upsilon^2}_{\mu G(e_2)}), \mathcal{W}\Omega({\upsilon^1}_{\mu G(e_1)})\right) + \sigma L_p^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)})$

Then, we have a $\mathfrak{f}_{ss}$ −sequence $\langle {\upsilon^m}_{\mu G(e_m)} \rangle$ in $\mathfrak{f}_{ss}(U)$ such that

a) ${\upsilon^{m+1}}_{\mu G(e_{m+1})} \mathrel{\tilde{\in}} \mathcal{W}\Omega({\upsilon^m}_{\mu G(e_m)}), {\upsilon^{m+2}}_{\mu G(e_{m+2})} \mathrel{\tilde{\in}} \mathcal{Z}\Omega({\upsilon^{m+1}}_{\mu G(e_{m+1})})$.

b) $L_p^b({\mathrm{u}^0}_{\mu G(\mathrm{e}_0)}, {\upsilon^1}_{\mu G(e_1)}) \mathrel{\tilde{>}} \tilde{0}$.

c) $\widetilde{\mathfrak{D}}_p^b\left({\upsilon^{m+1}}_{\mu G(e_{m+1})}, {\upsilon^m}_{\mu G(e_m)}\right) \mathrel{\tilde{\leqq}}$
$\mathcal{H}_p^b\left(\mathcal{Z}\Omega({\upsilon^m}_{\mu G(e_m)}), \mathcal{W}\Omega({\upsilon^{m-1}}_{\mu G(e_{m-1})})\right) + \sigma L_p^b({\upsilon^m}_{\mu G(e_m)}, {\upsilon^{m-1}}_{\mu G(e_{m-1})})$.

d) $\widetilde{\mathfrak{D}}_p^b\left({\upsilon^{m+2}}_{\mu G(e_{m+2})}, {\upsilon^{m+1}}_{\mu G(e_{m+1})}\right) \mathrel{\tilde{\leqq}} \mathcal{H}_p^b\left(\mathcal{Z}\Omega\left({\upsilon^{m+1}}_{\mu G(e_{m+1})}\right), \mathcal{W}\Omega({\upsilon^m}_{\mu G(e_m)})\right)$
$+\sigma L_p^b({\upsilon^{m+1}}_{\mu G(e_{m+1})}, {\upsilon^m}_{\mu G(e_m)})$.

Then we have, $\mathcal{H}_p^b\left(\mathcal{Z}\Omega({\upsilon^{m+1}}_{\mu G(e_{m+1})}), \mathcal{W}\Omega({\upsilon^m}_{\mu G(e_m)})\right) \mathrel{\tilde{\leqq}} \mathfrak{h} L_p^b({\upsilon^{m+1}}_{\mu G(e_{m+1})}, {\upsilon^m}_{\mu G(e_m)})$

Thus$\mathcal{H}_p^b\left(\mathcal{Z}\Omega({\upsilon^{m+1}}_{\mu G(e_{m+1})}), \mathcal{W}\Omega({\upsilon^m}_{\mu G(e_m)})\right) \mathrel{\tilde{\leqq}} \Re L_p^b({\upsilon^{m+1}}_{\mu G(e_{m+1})}, {\upsilon^m}_{\mu G(e_m)})$

where $\Re = \mathfrak{h} + \sigma < 1$

hence $\widetilde{\mathfrak{D}}_p^b\left({\upsilon^{m+1}}_{\mu G(e_{m+1})}, {\upsilon^m}_{\mu G(e_m)}\right) \mathrel{\tilde{\leqq}} \Re L_p^b({\upsilon^{m-1}}_{\mu G(e_{m-1})}, {\upsilon^m}_{\mu G(e_m)})$

$\lesseqqgtr \Re Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \mathcal{Z\Omega}\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}\right)\right),$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathcal{W\Omega}\left(\upsilon^m{}_{\mu G(e_m)}\right)\right)\}$

$\lesseqqgtr \Re Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)$.

Now, we have two cases:

First: if $Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)$

$=\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)$ then $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \upsilon^m{}_{\mu G(e_m)}\right) \lesseqqgtr \Re\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)$

$\widetilde{<} \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \upsilon^m{}_{\mu G(e_m)}\right)$ that is contradiction.

Second: if $Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)$

$=\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right)$ then $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \upsilon^m{}_{\mu G(e_m)}\right) \lesseqqgtr \Re\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^m{}_{\mu G(e_m)}\right)$

$$\lesseqqgtr \Re^2\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m-1}{}_{\mu G(e_{m-1})}, \upsilon^{m-2}{}_{\mu G(e_{m-2})}\right)$$

$$\lesseqqgtr \Re^n\widetilde{\mathfrak{D}}_p^b\left(\upsilon^1{}_{\mu G(e_1)}, \mathrm{u}^0{}_{\mu G(\mathrm{e}_0)}\right)$$

Since $\Re = \mathfrak{h} + \sigma < 1$ then $\langle \upsilon^m{}_{\mu G(e_m)} \rangle$ is $\mathfrak{f}_{ss} -$ Cauchy sequence

Since $\left(\mathrm{F}_{ss}(U), \widetilde{\mathfrak{D}}_p^b\right)$ be a complete F_{ss}- $b\mathscr{pms}$ then$\langle \upsilon^m{}_{\mu G(e_m)} \rangle$ is converge to $\mathrm{u}_{\mu G(\mathrm{e})}$

in$\mathrm{F}_{ss}(U)$, that is $\lim_{n\to\infty} \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathrm{u}_{\mu G(\mathrm{e})}\right) = \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{u}_{\mu G(\mathrm{e})}\right)$

since, $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+2}{}_{\mu G(e_{m+2})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr \mathcal{H}_p^b\left(\mathcal{Z\Omega}\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}\right), \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)$

$$\lesseqqgtr \Re L_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \mathrm{u}_{\mu G(\mathrm{e})}\right)$$

$\lesseqqgtr \Re Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \mathrm{u}_{\mu G(\mathrm{e})}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \mathcal{Z\Omega}\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}\right)\right),$

$$\widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)\}$$

By applying the triangular property of the $b\mathscr{pms}$,one get

$\widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr S[\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathrm{u}_{\mu G(\mathrm{e})}\right) + \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) - \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{u}_{\mu G(\mathrm{e})}\right)]$

$\lesseqqgtr S[\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathrm{u}_{\mu G(\mathrm{e})}\right) + \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)$

$\lim_{n\to\infty} \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr \Re\, \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)$

Then, $\widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) = \widetilde{0}$, hence $\mathrm{u}_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)$

That is $\mathrm{u}_{\mu G(\mathrm{e})}$ is $\mathfrak{f}_{ss}$ −fixed point for $\mathcal{W\Omega}$

Also, $\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+2}{}_{\mu G(e_{m+2})}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr \mathcal{H}_p^b\left(\mathcal{Z\Omega}\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}\right), \mathcal{W\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)$

$$\lesseqqgtr \Re L_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \mathrm{u}_{\mu G(\mathrm{e})}\right)$$

$\lesseqqgtr \Re Max\{\widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \mathrm{u}_{\mu G(\mathrm{e})}\right), \widetilde{\mathfrak{D}}_p^b\left(\upsilon^{m+1}{}_{\mu G(e_{m+1})}, \upsilon^{m+2}{}_{\mu G(e_{m+2})}\right),$

$\widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{W\Omega}\left(, \mathrm{u}_{\mu G(\mathrm{e})}\right)\right)\}$

$\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr S[\widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathrm{u}_{\mu G(\mathrm{e})}\right) + \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) - \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathrm{u}_{\mu G(\mathrm{e})}\right)]$

but $\lim_{n\to\infty} \widetilde{\mathfrak{D}}_p^b\left(\upsilon^m{}_{\mu G(e_m)}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) \lesseqqgtr \Re\, \widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right)$

Then, $\widetilde{\mathfrak{D}}_p^b\left(\mathrm{u}_{\mu G(\mathrm{e})}, \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)\right) = \widetilde{0}$, hence $\mathrm{u}_{\mu G(\mathrm{e})} \widetilde{\in} \mathcal{Z\Omega}\left(\mathrm{u}_{\mu G(\mathrm{e})}\right)$

That is $\mathrm{u}_{\mu G(\mathrm{e})}$ is $\mathfrak{f}_{ss}$ −fixed point for $\mathcal{Z\Omega}$

That is meanu$_{\mu G(\mathrm{e})}$ is common $\mathfrak{f}_{ss}$ −fixed point for $\mathcal{Z}\Omega$ and $\mathcal{W}\Omega$.

4. Conclusion

In this paper ,a new and flexible theoretical framework is studied that combines 𝕓-partial metric space and the concept of fuzzy soft set by introducing the definition of the 𝕓 -partial Hausdorff metric. In addition, some theorems related to the Banach principle were developed under new contractionary conditions .we generalized these theorem to the fuzzy soft 𝕓 -partial space ,proving the existence of common point for both single-valued and multi-valued $\mathfrak{f}_{ss}$ − mapping in this soft fuzzy space .Finally, this framework provides a bridge to many diverse applications ,as it is used in fuzzy dynamical systems and to improve many artificial intelligence algorithms.

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