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RESEARCH ARTICLE - MATHEMATICS

A novel structure for measurable functions sequence convergence based on approach measure space

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Abstract

In measure theory, there are several types of convergence of sequences like convergence in measure, almost everywhere and almost uniformly we presented definition of approach measure and approach measurable functions also we defined new concepts which convergence of sequences of approach measurable functions almost everywhere with respect approach measure and convergence of sequences of approach measurable functions in approach measure , we studied the relationship between them, We also raised the topic of "almost bounded" and what its relationship to "bounded"? We will discuss the relation between uniform convergence and convergence of sequences of approach measurable functions in approach measure, the relation between convergence of sequences of approach measurable functions almost everywhere with respect approach measure and convergence of sequences of approach measurable functions in approach measure.

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1. Introduction

Measure theory is one of the fundamental pillars of modern mathematics and extends to many different branches and applications in the natural and engineering sciences. Measure theory is an important topic in mathematical analysis. In 1966, Taylor [1] gave introduction to measure and integration. In 1989, Lowen [2] introduced common super category approach spaces of topological. In 1990, Dalang and Willinger [3] made clear no-arbitrage in stochastic securities markets models and equivalent martingale measures. In 1994, Baekeland and Lowen [4] explained separability and Lindelof in approach spaces. In 1997, Lowen [5] found relation between approach spaces and topology-uniformity-metric. In 2007, Boushra Y. Hussein [6] presented equivalent measure under the process is martingale. In 2010, Boushra Y. Hussein, AL-Ta. and AL-Mayahi [7] presented equivalent martingale measures on L^p $-$space, and in 2011 Boushra Y. Hussein [8] found equivalent measure which the process is (super/sub) martingale, then in 2020, Boushra Y. Hussein [9] introduced equivalent measure in ordered Banach algebra with locally Martingale of the deflator process. In 2021 and 2022, Abbas and Hussein [10] presented results on normed approach space and Banach space. In 2022, they gave results about topological approach vector space [11]. In 2022, Hussein and Abd [12] studied normed approach space. In 2022, Hussein, Huda and Wshayeh [13] defined on state space of measurable studied symmetric approach t^w $-$Banach algebra with some results. In 2022, Neamah and Hussein [14] presented the results of completion of t^w normed approach space when it's not contained identity.

In this paper we defined convergence almost everywhere by use approach measure structure. Also, we define convergence by approach measure then we find relationship between them.

This work is consisting four sections: In section 1, we presented to the reader Introduction. In section 2, we presented the most important definitions and concepts, in section 3 we introduced to new definitions and examples for our research which approach measure and finite approach measure. In Section 4, we gave some results about the two new concepts of the convergence.

2. Preliminary

We introduced main definitions related to subject in measure theory.

Definition 2.1 [15]
Let $\aleph$ be a non-empty set and F be a collection of subsets of $\aleph$, F is called $\sigma -$field if:
- $\aleph \in F$.
- If ß $\in F$, then ß$^c \in F$.
- If $A_1, A_2, \cdots, F$, we have $\bigcup_{n=1}^{\infty} A_n \in F$.

A measurable space is a pair $(\aleph, F)$.

Definition 2.2 [15]
If (Ω, F) measurable space. A set function μ: $F \rightarrow [0, \infty]$ is called measure on (Ω, F), if:
- $\mu(\emptyset) = 0$.
- $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty}(\mu A_n)$, where $\{A_n\}$ sequence of disjoint sets in F.

A measure space is a triple (Ω, F, μ).

Definition 2.3 [15]
Let (Ω, F, μ) be a measure space. A property of points Ω is said to hold $\mu -$almost everywhere if there exist Ţ $\in F$ such that μ (Ţ) $= 0$ which contains every point that the property fails to hold.

Definition 2.4 [15]
Let $(\aleph, F, \mu)$ be measure space, the sequence $\{♀_n\}_{n=1}^{\infty}$ of real valued functions on $\aleph$ is convergent almost everywhere to real valued function ♀ on $\aleph$ if there is a set Đ $\subseteq \aleph$ such that $♀_n(x) \rightarrow ♀(x)$ for each $x \notin$ Đ and Đ $\subseteq$ Ꚅ $\in F$ with μ (Ꚅ) $= 0$.

Definition 2.5 [16]
Let $(\aleph, d)$ be metric space , the sequence $\{♀_n\}_{n=1}^{\infty}$ of real valued functions on $\aleph$ is uniformly convergent to real valued function ♀ on $\aleph$ if for each $\epsilon > 0$ there exist Ǩ > 0 such that $|♀_n(x) - ♀(x)\ | < \epsilon$ for each $x \in \aleph$, $n >$ Ǩ (write $♀_n \rightarrow^u ♀$).

Definition 2.6 [16]
Let $(\aleph, d)$ be metric space, the sequence $\{♀_n\}_{n=1}^{\infty}$ of real valued functions on $\aleph$ is bounded if there exist Ţ > 0 such that $|♀_n(x)| \leq$ Ţ for each $x \in \aleph$.

Definition 2.7 [6]
If X a non-empty set, a function $\delta: X \times 2^X \rightarrow [0, \infty]$ is a distance on X if every $x \in X$, for all $A \in 2^X$, the following conditions are met:
- $x \in A \Rightarrow \delta(x,\ A) = 0$.
- For all $B \in 2^X$: $A \subset B \Rightarrow \delta(x,\ B) \leq \delta(x,\ A)$.
- For all $A_1, A_2 \in 2^X, \delta(x, A_1 \cup\ A_2) = min_{A_i \in \psi}[\delta(x,\ A_1), \delta(x,\ A_2)]$
- For all $B \in 2^u$: $\delta(u, A) \leq \delta(u, B) + sup_{s \in B}\ \delta(s, A)$.

$\Re_\delta(x) = \{\vartheta \in [0, \infty]^X |\ inf_{a \in A}\ \vartheta(a) \leq \delta(a,\ A)\}$ is called an approach system, the pair $(X, \Re_\delta(x))$ is called approch space and denoted by app-spaces.

Theorem 2.1 [16]
Every convergent sequence is bounded.

3. Approach measure and finite approach measure

Definition 3.1
If $\aleph$ is a non-empty set and F $\sigma -$filed. A function $\mu_\delta: F \times 2^\aleph \rightarrow [0, \infty]$ is called approach measure on $\aleph$ (denoted by AM) if the following conditions are met:
- $\mu_\delta(\emptyset, ß) = 0$, ß $\in 2^\aleph$.
- μ_δ (Ą, ß) $= \infty$ if Ą $=$ ß^c, Ą $\in$ F, ß $\in 2^X$, Ą $\neq \emptyset$.
- μ_δ $(\bigcup_{n=1}^{\infty}$ Ą$_n$, ß$) = \sum_{n=1}^{\infty} \mu_\delta($Ą$_n$, ß$)$ where $\{A_n\}$ is sequence of disjoint sets in F.

The triple $(\aleph, F, \mu_\delta)$ is said to be approach measure space and denoted by AMS.

Definition 3.2

Let $\aleph$ be a non-empty set and F be $\sigma-$ filed, if μ_δ is AM on $\aleph$, we call μ_δ is finite approach measure (denote by FAM) if for each $Ą \neq ß^c$ we have $\mu_\delta(Ą, ß) \neq \infty, Ą \in F, ß \in 2^X$.

Example 3.1

Let $\aleph$ be a non-empty set and F be a $\sigma-$filed, a function $\mu_\delta : F \times 2^\aleph \to [0, \infty]$ is defined by:

$$\mu_\delta\,(Ą, ß) = \begin{cases} 0, & Ą = \emptyset \\ \infty, & Ą \neq \emptyset \end{cases}$$

Where $Ą \in F, ß \in 2^\aleph$, this function μ_δ is not finite approach measure.

Sol:

Let $\aleph = \{a.b, c\}, F = 2^\aleph$ and μ_δ which is defined as above. If we choose $Ą = \{a\} \in F, ß = \{a, c\} \in 2^\aleph$ it is clear $Ą \neq ß^c$. By definition of the function (μ_δ) we have $\mu_\delta\,(Ą, ß) = \infty$.

So, we get $Ą \neq ß^c$ but $\mu_\delta\,(Ą, ß) = \infty, Ą \in F, ß \in 2^\aleph$. μ_δ is not finite approach measure.

Proposition 3.1

Let $(\aleph, F, \mu_\delta)$ be AMS, if $Ą \subseteq ß$ then $\mu_\delta(Ą, C) \leq \mu_\delta(ß, C)$, where $Ą, ß \in F, C \in 2^\aleph$.

Proposition 3.2

Let $(\aleph, F, \mu_\delta)$ be AMS, if $Ɓ \subseteq Ą \in F$ then $\mu_\delta(Ą - Ɓ, C) = \mu_\delta(Ą, C) - \mu_\delta(Ɓ, C)$ where $Ą, Ɓ \in F, C \in 2^\aleph$.

Proposition 3.3

Let $(\aleph, F, \mu_\delta)$ be AMS and $\{Ą_k\}_{k=1}^\infty$ be increasing sequence of sets of F, then $\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \lim_{k\to\infty} \mu_\delta(Ą_k, ß), ß \in 2^\aleph$.

Proof

Let $\check{G}_1 = Ą_1, \check{G}_i = Ą_i - Ą_{i-1}, i > 1$. It is clear $\cap_{i=1}^k \check{G}_i = \emptyset$.

$\cup_{i=1}^k \check{G}_i = Ą_1 \cup (Ą_2 - Ą_1) \cup (Ą_3 - Ą_2) \cup \cdots \cup (Ą_{k-2} - Ą_{k-1}) \cup (Ą_{k-1} - Ą_k)$

$$\cup_{i=1}^k \check{G}_i = Ą_k \tag{1}$$

We conclude $\cup_{i=1}^k \check{G}_i = \cup_{i=1}^k Ą_k$.

Let $ß \in 2^\aleph$:

$$\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \mu_\delta(\cup_{i=1}^k \check{G}_i, ß)$$
$$\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \textstyle\sum_{i=1}^k \mu_\delta(\check{G}_i, ß)$$
$$\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \lim_{k\to\infty} \textstyle\sum_{i=1}^k \mu_\delta(\check{G}_i, ß)$$
$$\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \lim_{k\to\infty} \mu_\delta\,(\cup_{i=1}^k \check{G}_i, ß)$$

By (1) we can get:

$$\mu_\delta(\cup_{k=1}^\infty Ą_k, ß) = \lim_{k\to\infty} \mu_\delta\,(Ą_k, ß) \ \blacksquare$$

Proposition 3.4

Let $(\aleph, F, \mu_\delta)$ be AMS and $\{Ą_k\}_{k=1}^\infty$ be decreasing sequence of sets of F, if μ_δ is FAM then $\mu_\delta(\cap_{k=1}^\infty Ą_k, ß) = \lim_{k\to\infty} \mu_\delta(Ą_k, ß)$ where $A_k \neq ß^c, ß \in 2^\aleph$.

Proof

Let $C_k = A_1 - A_k$, $k = 1,2,\cdots$. Since $\{A_k\}_{k=1}^{\infty}$ is decreasing sequence, so that $A_1 \supset A_2 \supset A_3 \supset \cdots$. Therefor $C_1 \subset C_2 \subset C_3 \subset \cdots$, so that $\{C_k\}_{k=1}^{\infty}$ is increasing sequence.

$\cup_{k=1}^{\infty} C_k = \cup_{k=1}^{\infty} (A_1 - A_k)$
$\cup_{k=1}^{\infty} C_k = A_1 - \cap_{k=1}^{\infty} A_k$

Let $ß \in 2^{\aleph}$, we will choose $A_k \neq ß^c$.
$\mu_\delta(\cup_{k=1}^{\infty} C_k, ß) = \mu_\delta(A_1 - \cap_{k=1}^{\infty} A_k, ß)$

By Proposition 3.3, we get:

$\mu_\delta(\cup_{k=1}^{\infty} C_k, ß) = lim_{k\to\infty}\ \mu_\delta(C_k, ß)$
$\mu_\delta(A_1 - \cap_{k=1}^{\infty} A_k, ß) = \lim\limits_{k\to\infty} \mu_\delta(C_k, ß)$
$\mu_\delta(A_1 - \cap_{k=1}^{\infty} A_k, ß) = \lim\limits_{k\to\infty} \mu_\delta(A_1 - A_k, ß)$

By Proposition 3.3, we get:

$\mu_\delta(A_1, ß) - \mu_\delta(\cap_{k=1}^{\infty} A_k, ß) = \lim\limits_{k\to\infty} \mu_\delta(A_1, ß) - \lim\limits_{k\to\infty} \mu_\delta(A_k, ß)$
$\mu_\delta(A_1, ß) - \mu_\delta(\cap_{k=1}^{\infty} A_k, ß) = \mu_\delta(A_1, ß) - \lim\limits_{k\to\infty} \mu_\delta(A_k, ß) \qquad (2)$

Since μ_δ is FAM and $A_k \neq ß^c$, $k = 1,2,\cdots$, then $\mu_\delta(A_1, ß) \neq \infty$.
So, we can cancel $\mu_\delta(A_1, ß)$ from both sides on (1), we get:

$\mu_\delta(\cap_{k=1}^{\infty} A_k, ß) = \lim\limits_{k\to\infty} \mu_\delta(A_k, ß)$ ■

4. Convergence of sequences of approach measurable functions in approach measure

In this section we will present the main results.

Definition 4.1

Let $(\aleph_1, F_1, \mu_{\delta_1})$, $(\aleph_2, F_2, \mu_{\delta_2})$ be two AMS. A function $\text{♀}: (\aleph_1, F_1, \mu_{\delta_1}) \to (\aleph_2, F_2, \mu_{\delta_2})$ is called approach measurable function if satisfies the following conditions:

- $\text{♀}(A) \in F_2$ for each $A \in F_1$.
- $\mu_{\delta_1}(A, B) \leq \mu_{\delta_2}(\text{♀}(A), \text{♀}(B))$ for each $A \in F_1, B \in 2^{\aleph}$.
-

Definition 4.2

Let $(\aleph, F, \mu_\delta)$ be AMS and $\text{♀}, \text{♀}_1, \text{♀}_2, \cdots,$ be real valued approach measurable functions, the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is convergent almost everywhere to ♀ with respect to approach measure if there is a set $Đ \in F$ such that $\text{♀}_n(x) \to \text{♀}(x)$ for each $x \notin Đ$ and $Đ \subseteq Ɗ \in F$ with $\mu_\delta(Ɗ, ß) = 0$ for each $Ɗ \neq ß^c$, $ß \in 2^{\aleph}$.

Definition 4.3

Let $(\aleph, F, \mu_\delta)$ be AMS and $\text{♀}, \text{♀}_1, \text{♀}_2, \cdots,$ be real valued approach measurable functions, the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is convergent to ♀ in approach measure if for each $\epsilon > 0$

$$\lim_n \mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\}, ß) = 0,$$

for some $ß \in 2^{\aleph}$.
We write $\text{♀}_n \to \text{♀}$ in AM.

Definition 4.4
Let $(\aleph, F, \mu_\delta)$ be AMS and $\text{♀}, \text{♀}_1, \text{♀}_2, \cdots,$ be real valued approach measurable functions, the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded with respect to approach measure if for each $\epsilon > 0$ there exist $Ţ > 0$ such that $\mu_\delta(\{x \in \aleph, |\text{♀}_n(x)| > Ţ\}, ß) < \epsilon$ for some $ß \in 2^{\aleph}$.

Definition 4.5
Let $(\aleph, F, \mu_\delta)$ be AMS and $\text{♀}, \text{♀}_1, \text{♀}_2, \cdots,$ be real valued approach measurable functions, the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost uniformly convergent to ♀ in approach measure if for each $\epsilon > 0$ there is a set $Ą \in F$ such that $\mu_\delta(Ą, ß) < \epsilon$ for some $ß \in 2^{\aleph}$ and $\text{♀}_n \rightarrow^{u} \text{♀}$ on $Ą^{C}$.

Theorem 4.1
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, $\text{♀}_1, \text{♀}_2, \ldots$ be real valued approach measurable functions on $\aleph$, if $\text{♀}_n \rightarrow^{u} \text{♀}$ then $\text{♀}_n \rightarrow \text{♀}$ in AM.

Proof
Suppose $\text{♀}_n \rightarrow^{u} \text{♀}$, so that for each $\epsilon > 0$ there exist $\check{K} > 0$ such that $|\text{♀}_n(x) - \text{♀}(x)| < \epsilon$ for each $x \in \aleph$, $n > \check{K}$, then there is not element in$\aleph$ satisfies $|\text{♀}_n(x) - \text{♀}(x)| < \epsilon$, $n > \check{K}$, there for $\left\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\right\} = \emptyset$, $n > \check{K}$.

$\mu_\delta\{x \in \aleph, |\text{♀}_n(x)| > \epsilon\}, ß) = 0$ *for each* $ß \in 2^{\aleph}, n > \check{K}$.
$\lim_{n} \mu_\delta(\left\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\right\}, ß) = 0$ *for each* $ß \in 2^{\aleph}$,

Therefor $\text{♀}_n \rightarrow \text{♀}$ in AM ■.

Theorem 4.2
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, $\text{♀}_1, \text{♀}_2, \ldots$ be real valued approach measurable functions on $\aleph$, if $\{\text{♀}_n\}_{n=1}^{\infty}$ is bounded then $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded with respect to approach measure.

Proof
Since $\{\text{♀}_n\}_{n=1}^{\infty}$ is bounded then there exist $Ţ > 0$ such that $|\text{♀}_n(x)| \leq Ţ$ for each $x \in \aleph$.
So that $\{x \in \aleph, |\text{♀}_n(x)| > Ţ\} = \emptyset$.
$\mu_\delta\{x \in \aleph, |\text{♀}_n(x)| > Ţ\}, ß) = 0$ *for ezch* $ß \in 2^{\aleph}$
$\mu_\delta\{x \in \aleph, |\text{♀}_n(x)| > Ţ\}, ß) < \epsilon$ *for each* $ß \in 2^{\aleph}$

Therefor $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded with respect to approach measure ■.

Theorem 4.3
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, $\text{♀}_1, \text{♀}_2, \ldots$ be real valued approach measurable functions on $\aleph$, if $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded with respect to approach measure and $\text{♀}_n \rightarrow \text{♀}$ in AM then ♀ is almost bounded with respect to approach measure.

Proof
Suppose $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded and $\text{♀}_n \rightarrow \text{♀}$ in AM
Since $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost bounded then for each $\rho > 0$, there exist $Ţ > 0$ such that

$$\mu_\delta\{x \in \aleph, |\text{♀}_n(x)| > Ţ\}, ß) < \frac{\rho}{2} \textit{ for some } ß \in 2^{\aleph}$$

Since $\text{♀}_n \rightarrow \text{♀}$ in AM then:
For each $\epsilon > 0$, $\lim_n \mu_\delta(\left\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\right\}, ß) = 0$ for some $ß \in 2^{\aleph}$.
If $\epsilon = 1$, $\mu_\delta(\{x \in \aleph, |\text{♀}(x) - \text{♀}_n(x)| > 1\}, ß) < \frac{\rho}{2}$ for some $ß \in 2^{\aleph}$.
Now we must proof $\mu_\delta\{x \in \aleph, |\text{♀}(x)| > Ţ + 1\}, ß) < \rho$ for some $ß \in 2^{\aleph}$.

$$\{x \in \aleph, |Ϙ(x)| > Ţ + 1\} = \{x \in \aleph, |Ϙ(x) - Ϙ_n(x) + Ϙ_n(x)| > Ţ + 1\} \tag{3}$$

We claim $\{x \in \aleph, |Ϙ(x)| > Ţ + 1\} \subset \{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\} \cup \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$.
Let $ȥ \in \{x \in \aleph, |Ϙ(x)| > Ţ + 1\}$, by (3) we get:

$$|Ϙ(ȥ) - Ϙ_n(ȥ) + Ϙ_n(ȥ)| > Ţ + 1$$

$$|Ϙ(ȥ) - Ϙ_n(ȥ)| + |Ϙ_n(ȥ)| \geq |Ϙ(ȥ) - Ϙ_n(ȥ) + Ϙ_n(ȥ)| > Ţ + 1 \tag{4}$$

If $ȥ \notin \{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\}$
We must proof $ȥ \in \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$

$$|Ϙ(ȥ) - Ϙ_n(ȥ)| \leq 1$$

Then $1 + |Ϙ_n(ȥ)| > Ţ + 1$ (by (4))
$|Ϙ_n(ȥ)| > Ţ$
There for $ȥ \in \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$
Now If $ȥ \notin \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$
$|Ϙ_n(ȥ)| \leq Ţ$
Then $|Ϙ(x) - Ϙ_n(x)| + Ţ > Ţ + 1$ (by (4))

$$|Ϙ(x) - Ϙ_n(x)| > 1$$

Therefor $ȥ \in \{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\}$
$ȥ \in \{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\} \cup \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$
so, we get:

$\{x \in \aleph, |Ϙ(x)| > Ţ + 1\} \subset \{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\} \cup \{x \in \aleph, |Ϙ_n(x)| > Ţ\}$
$\mu_\delta\{x \in \aleph, |Ϙ(x)| > Ţ + 1\}, ß) \leq \mu_\delta(\{x \in \aleph, |Ϙ(x) - Ϙ_n(x)| > 1\}, ß) + \mu_\delta\{x \in \aleph, |Ϙ(x)| > Ţ + 1\}, ß)$ for some $ß \in 2^{\aleph}$
$\mu_\delta\{x \in \aleph, |Ϙ(x)| > Ţ + 1\}, ß) \leq \frac{\rho}{2} + \frac{\rho}{2}$
$\mu_\delta\{x \in \aleph, |Ϙ(x)| > Ţ + 1\}, ß) \leq \rho$ for some $ß \in 2^{\aleph}$.

Then Ϙ is almost bounded with respect to approach measure ■.

Theorem 4.4
Let $(\aleph, F, \mu_\delta)$ be AMS and $Ϙ, Ϙ_1, Ϙ_2, \ldots$ be real valued approach measurable functions on $\aleph$, if μ_δ is FAM and $\{Ϙ_n\}_{n=1}^{\infty}$ is convergent almost everywhere to Ϙ with respect to approach measure then $Ϙ_n \to Ϙ$ in AM.

Proof
Suppose $\{Ϙ_n\}_{n=1}^{\infty}$ is convergent almost everywhere to Ϙ with respect to Approach Measure.
Now we must proof $Ϙ_n \to Ϙ$ in AM.
Let $\epsilon > 0$
Define $Ţ_n = \{x \in \aleph, |Ϙ_n(x) - Ϙ(x)| > \epsilon\}$
Let $Ú_n = \cup_{n=k}^{\infty} Ţ_k$
So that:

$Ú_n = Ţ_n \cup Ţ_{n+1} \cup Ţ_{n+2} \cup \ldots$
$Ú_{n+1} = Ţ_{n+1} \cup Ţ_{n+2} \cup Ţ_{n+2} \cup \ldots$
$Ú_{n+2} = Ţ_{n+2} \cup Ţ_{n+3} \cup Ţ_{n+3} \cup \ldots$

Thus, we conclude $Ú_n \supset Ú_{n+1} \supset Ú_{n+2} \supset \ldots$
Then $\{Ú_n\}_{n=1}^{\infty}$ is decreasing sequence.
The set $\{x \in \aleph : \text{♀}_n(x)$ does not converges to $\text{♀}(x)\}$ which contains the set $\cap_n Ú_n$
Since $\{\text{♀}_n\}_{n=1}^{\infty}$ convergent almost everywhere to ♀ with respect to Approach Measure.
So $\mu_\delta(\cap_n Ú_n, ß) = 0$ for each $ß^c \neq \cap_{n=k}^{\infty} Ú_n \in 2^{\aleph}$.
By Proposition 3.7 we get:
$\mu_\delta(\cap_n Ú_n, ß) = \lim_{n\to\infty} \mu_\delta(Ú_n, ß) = 0$ where $A_k \neq ß^c$, $ß \in 2^{\aleph}$
Since $Ʈ_n \subseteq \cup_{n=k}^{\infty} Ʈ_k$, then $Ʈ_n \subseteq Ú_n$
We get $\lim_n \mu_\delta(Ʈ_n, ß) \leq \lim_n \mu_\delta(Ú_n, ß)$ by Proposition 3.1.
So, we get $\lim_n \mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\}, ß) = 0$ for some $ß \in 2^{\aleph}$
Then we have proven $\text{♀}_n \to \text{♀}$ in AM ■.

Theorem 4.5
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, $\text{♀}_1, \text{♀}_2, \ldots$ be real valued approach measurable functions on $\aleph$, if the sequence $\text{♀}_n \to \text{♀}$ in AM, then there is a sub sequence of $\{\text{♀}_n\}_{n=1}^{\infty}$ that convergent to ♀ almost everywhere with respect to Approach Measure.

Proof
Suppose $\text{♀}_n \to \text{♀}$ in AM
So that for each $\epsilon > 0$ we have:

$\lim_n \mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\}, ß) = 0$ for some $ß \in 2^{\aleph}$
$\mu_\delta(\{x \in \aleph, |\text{♀}_{n_1}(x) - \text{♀}(x)| > 1\}, ß) \leq \frac{1}{2}$ for some $ß \in 2^{\aleph}$
$\mu_\delta(\{x \in \aleph : |\text{♀}_{n_2}(x) - \text{♀}(x)| > \frac{1}{2}\}, ß) \leq \frac{1}{4}$ for some $ß \in 2^{\aleph}$
$\mu_\delta(\{x \in \aleph : |\text{♀}_{n_3}(x) - \text{♀}(x)| > \frac{1}{3}\}, ß) \leq \frac{1}{8}$ for some $ß \in 2^{\aleph}$

Until we arrive:

$$\mu_\delta(\{x \in \aleph : |\text{♀}_{n_k}(x) - \text{♀}(x)| > \tfrac{1}{k}\}, ß) \leq \tfrac{1}{2^k} \text{ for some } ß \in 2^{\aleph}, k=1,2,... \tag{5}$$

Let $Ɲ_k = \{x \in \aleph : |\text{♀}_{n_k}(x) - \text{♀}(x)| > \frac{1}{k}\}$
Let $y \notin \cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k$
Then $y \notin \cup_{k=j}^{\infty} Ɲ_k$
Therefor $|\text{♀}_{n_k}(y) - \text{♀}(y)| > \frac{1}{k}$, k=j, j+1, j+2, …
So $\{\text{♀}_{n_k}\}_{n=1}^{\infty}$ converges to ♀ at each $y \notin \cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k$
Now we must proof $\mu_\delta(\cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k, ß) = 0$ where $\cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k \neq ß^c$
Let $ß \in 2^{\aleph}$
We will expect $\cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k = ß^c$
To avoid values that make $\mu_\delta(\cap_{j=1}^{n} \cup_{k=j}^{\infty} Ɲ_k, ß) = \infty$
Since $\mu_\delta(\cup_{k=j}^{\infty} Ɲ_k, ß) \leq \sum_{k=j}^{\infty} \mu_S(Ɲ_k, ß)$

$\mu_\delta(\cup_{k=j}^{\infty} Ɲ_k, ß) \leq \sum_{k=j}^{\infty} \frac{1}{2^k}$

Since $\sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^j}}{1-\frac{1}{2}} = \frac{1}{2^{j-1}}$
Then $\mu_\delta(\cup_{k=j}^{\infty} Ɲ_k, ß) \leq \frac{1}{2^{j-1}}$

There for μ_δ $(\bigcup_{k=j}^{\infty} \mathrm{N}_k$,ß) = 0 when j converges to ∞
μ_δ $(\bigcap_{j=1}^{n} \bigcup_{k=j}^{\infty} \mathrm{N}_k$,ß) = 0 where $\bigcap_{j=1}^{n} \bigcup_{k=j}^{\infty} \mathrm{N}_k \neq$ ß^c
So $\{\text{♀}_{n_k}\}_{n=1}^{\infty}$ is a sub sequence of $\{\text{♀}_n\}_{n=1}^{\infty}$ that convergent to ♀ almost everywhere with respect to approach measure ■.

Theorem 4.6
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, ♀_1,♀_2, … be real valued approach measurable on $\aleph$ if μ_δ is FAM and each subsequence of $\{\text{♀}_n\}_{n=1}^{\infty}$ has a subsequence that converges to ♀ almost everywhere with respect to approach measure then $\text{♀}_n \to$ ♀ in AM.

Proof
Suppose each subsequence of $\{\text{♀}_n\}_{n=1}^{\infty}$ has a subsequence that converges to ♀ almost everywhere with respect to approach measure.
We must proof $\text{♀}_n \to$ ♀ in AM.
Suppose ♀_n is not convergence to ♀ in AM.
So there exist $\epsilon > 0$ such that $\lim_n$ $\mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \epsilon\}$, ß) = k > 0 for each ß $\in 2^{\aleph}$.
We can choose sub sequence $\{\text{♀}_{n_k}\}_k$ such that:

$\lim_n$ $\mu_\delta(\{x \in \aleph, |\text{♀}_{n_k}(x) - \text{♀}(x)| > \epsilon\}$, ß) = k > 0 for each ß $\in 2^{\aleph}$

Let $\{\text{♀}_{n_{k_j}}\}_j$ subsequence of $\{\text{♀}_{n_k}\}_k$.
Then $\{\text{♀}_{n_{k_j}}\}_j$ converges to ♀ almost everywhere with respect to approach measure (by hypothesis).
Therefor there exist Đ ∈ F such that $\text{♀}_{n_{k_j}}(x) \to$ ♀ (x) for each x ∉ Đ.
And $\mu_\delta(\{x \in \aleph, |\text{♀}_{n_{k_j}}(x) - \text{♀}(x)| > \epsilon\}$, ß) = 0 for each $\{x \in \aleph, |\text{♀}_{n_{k_j}}(x) - \text{♀}(x)| > \epsilon\} \neq$ ß^c …
But by $\lim_n$ $\mu_\delta(\{x \in \aleph, |\text{♀}_{n_k}(x) - \text{♀}(x)| > \epsilon\}$, ß) = k > 0 for each ß $\in 2^{\aleph}$.
This is a contradiction. Then $\text{♀}_n \to$ ♀ in AM.

Theorem 4.7
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, ♀_1,♀_2, … be real valued approach measurable functions on $\aleph$, if the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost uniformly convergent to ♀ in approach measure, then $\text{♀}_n \to$ ♀ in AM.

Proof
Suppose $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost uniformly convergent to ♀ in approach measure.
For each $\epsilon > 0$ there is a set Ą ∈ F such that μ_δ (Ą , ß) $< \epsilon$ for some ß $\in 2^{\aleph}$ and $\text{♀}_n \to^{u}$ ♀ on Ą^C.
So that there exist Ǩ > 0 such that $|\text{♀}_n(x) - \text{♀}(x)| < \rho$ for each x ∈ Ą^C, n>Ǩ, $\rho > 0$.

$\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \rho\} \subseteq$ Ą
$\mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \rho\}$, ß) $\leq$ μ_δ (Ą , ß) $< \epsilon$ for some ß $\in 2^{\aleph}$
$\lim_n$ $\mu_\delta(\{x \in \aleph, |\text{♀}_n(x) - \text{♀}(x)| > \rho\}$, ß) = 0 for some ß $\in 2^{\aleph}$

Therefor $\text{♀}_n \to$ ♀ in AM ■.

Theorem 4.8
Let $(\aleph, F, \mu_\delta)$ be AMS and ♀, ♀_1,♀_2, … be real valued approach measurable functions on $\aleph$, if the sequence $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost uniformly convergent to ♀ in approach measure then $\text{♀}_n \to$ ♀ almost everywhere with respect to approach measure.

Proof
Suppose $\{\text{♀}_n\}_{n=1}^{\infty}$ is almost uniformly convergent to ♀ in approach measure.

For each positive integer number n.
Let $A_n \in F$
Put A_n with $\mu_\delta (A_n , ß) < \frac{1}{n}$ for some $ß \in 2^{\aleph}$ and $\varphi_n \to^{u} \varphi$ on A_n^C
Let $\check{G} = \cup_{k=1}^{\infty} A_n^C$
Since $\varphi_n \to^{u} \varphi$ on A_n^C then $\varphi_n \to^{u} \varphi$ on $\check{G}$.

$\mu_\delta (\check{G}^C , ß) = \mu_\delta (\cap_{k=1}^{\infty} A_n, ß)$ for some $ß \in 2^{\aleph}$
$\mu_\delta (\check{G}^C , ß) \leq \mu_\delta (A_n , ß) < \frac{1}{n}$ for some $ß \in 2^{\aleph}$
$\mu_\delta (\check{G}^C , ß) = 0$ as $n \to \infty$ for some $ß \in 2^{\aleph}$

Therefor $\varphi_n \to \varphi$ almost everywhere with respect to approach measure■.

5. Conclusions

This research is to create a new structure for the concept of measurement that differs from classical measurement and new concepts which convergence of sequence almost everywhere with respect to approach measure, convergence in approach measure, convergence almost uniformly with respect to approach measure, classical uniform convergence. the results were:

- Every sequence which uniform convergence is convergence in AM.
- Sequence of real valued approach measurable functions which is almost bounded with respect to approach measure and convergence to approach measurable function, this function is also almost bounded with respect to approach measure.
- Every sequence which convergence almost everywhere with respect to approach measure and approach measure is finite, this sequence is convergence in AM.
- Every sequence which convergence in AM has a sub sequence that convergence almost everywhere with respect to approach measure.
- Every sequence which almost uniformly convergence in approach measure is convergence in AM.
- Every sequence which almost uniformly convergence in approach measure is convergence almost everywhere with respect to approach measure.

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