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RESEARCH ARTICLE - MATHEMATICS

The Hybrid Mathematical Model for Interpolation and Prediction Based on FBGS Algorithm to Determine Optimal Weights with Real Data Application

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Abstract

This study aims to improve prediction accuracy by applying a hybrid method based on BFGS algorithm to select optimal weights that the best evaluate the solution weight at each point. Newton and Lagrange's interpolation methods were used to construct the proposed hybrid model structure. The performance of the three methods (the basic methods and hybrid method) was evaluated through mean square error analysis. The results, applied to real data representing the radon gas spread rate in Baghdad, showed that the hybrid method clearly outperformed traditional methods, achieving a mean square error of 0.001501, compared to 0.00876 for Newton method and 0.003216 for Lagrange method. This approach contributes to improving prediction accuracy and reducing error by dynamically distributing weights based on BFGS algorithm. This study a solid foundation for developing more efficient models in various fields, such as temporal data forecasting and dynamic systems analysis.

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Keywords: Mathematical interpolation, BFGS algorithm, Hybrid method, accuracy improvement, MSE.

1. Introduction

Mathematical modeling is the process of constructing mathematical equations that describe the behavior of a given system using mathematical and logical relationships. It is used to understand natural phenomena and predict future values based on specific data. Mathematical interpolation models are among the fundamental models used to estimate values between known data points, helping solve engineering, physical, and economic problems.

The role of mathematical models in interpolation is to find a continuous mathematical relationship between a set of pre-specified points. When incomplete point data is provided, the goal is to create a model that allows values between those points to be estimated. The most prominent methods used for this are Newton's method and Lagrange method, which rely on polynomials to represent data. In mathematical forecasting, models rely on interpolation functions to predict future values based on current data. The accuracy of models is a critical factor in applications such as time-series data analysis and dynamic systems. Therefore, traditional methods are combined with modern optimization algorithms such as BFGS to improve the quality of results.

The Lagrange interpolation method is a mathematical interpolation technique used to estimate values between a set of known points. This method relies on constructing a polynomial that process through all the given points, making it useful for approximating mathematical functions based on specific data. This method has been relied upon in many studies. For example: a study used it to develop advanced theories for analyzing composite beams, panels, and shells, which led to improved accuracy compared for traditional models in the analysis of composite materials [1], also used to combined with Eulerian Lagrange approach was used to model material flow in double friction stir welding between aluminum and copper, which helped to accurately analyze the mechanical interaction between different materials [2], other study developed it to describe a flow dynamics in industrial applications [3], also used to improve constrained algorithms for optimizing composite geometric designs [4], and used to study the effect of gap size on the cavitation phenomenon in end-leakage vortices [5].

Newton's interpolation method is a numerical method used to construct a polynomial passing through a set of known points. This method is based on the concept of divided differences, making it more efficient when adding new points without having to recalculate all terms, as is the case with Lagrange method. This method has been relied upon in many studies. For example, a study presented a visual analysis of Newton's method using fractional derivatives and various iterations [6], a quantum version of Newton's method was developed for solving system of nonlinear equations [7], also used to solve linear inverse problems using neural network encoders [8], and was applied to solve problems with nonlinear boundary values [9], also other studies as [10],[11].

This study aims to create a hybrid mathematical model based on the BFGS algorithm for selecting optimal weights, capable of improving the prediction accuracy of mathematical model solution. Traditional methods such as Newton's interpolation and Lagrange interpolation suffer from some limitations, such as increased computational complexity when dealing with large datasets. Combining them with optimization algorithms such as BFGS helps reduce error and improve computational performance, making models more efficient and flexible in handling real-world data.

2. The Hybrid Mathematical (NL) Model for Interpolation and Prediction

The hybrid NL method combines the Newton's and Lagrange interpolation methods to improve the accuracy of estimates and reduce error. this is achieved by assigning optimal weights to each method based on BFGS algorithm, which optimizes the weight distribution to achieve the best possible results. The basic idea of hybrid NL method is to combine the advantages of Newton's interpolation, which is efficient when adding new points, and Lagrange interpolation, which provides accurate solutions when dealing with specific data. The weights w_1 and w_2 are dynamically adjusted so that the condition $w_1 + w_2 = 1$ is met, allowing for improved results and a significant reduction in the mean square error (MSE) compared to traditional methods. When building a hybrid model of mathematical integration, the following steps are followed:

1. Data input: determine the previously known set of points (x_i, y_i).
2. Calculate Newton's interpolation ($N(x,y)$) estimates: using Newton's interpolation method to calculate the estimated values of $N(x,y)$ using the equation below [12] [13]:
3.

$$N(x,y) = P_N(x) = f[x_0] + (x - x_0)f[x_0,x_1] + (x - x_0)(x - x_1)f[x_0,x_1,x_2] + \cdots + (x - x_0)(x - x_1)\dots(x - x_{n-1})f[x_0,x_1,x_2,\dots,x_n] \tag{1}$$

where $f[x_0,x_1,x_2,\dots,x_n]$ are the divided differences, defined recursively as:

$$f[x_i] = f(x_i) \tag{2}$$

$$f[x_i,x_{i+1},\dots,x_{i+k}] = \frac{f[x_{i+1},\dots,x_{i+k}] - f[x_i,\dots,x_{i+k-1}]}{x_{i+k} - x_i} \tag{3}$$

4. Calculate Lagrange interpolation ($L(x,y)$) estimates: apply Lagrange interpolation method to estimated values of $L(x,y)$ using the equation below [14] [15]:

$$L(x,y) = P_N(x) = \sum_{i=1}^{n} f(x_i)L_i(x) \tag{4}$$

where $L_i(x)$ are the Lagrange basis polynomials, defined as:

$$L_i(x) = \prod_{\substack{0<j<n \\ i \neq j}} \frac{x - x_i}{x - x_j} \tag{5}$$

5. Initialize initial weights: randomly assign initial values to the weights w_1 and w_2.
6. Build the hybrid model using the following equation:

$$Hybrid = w_1 N(x,y) + w_2 L(x,y) \tag{6}$$

7. Calculate MSE between actual and estimated values using equation:

$$MSE = \frac{1}{T}\sum_{i=1}^{T}(y_i - Hybrid_i)^2 \tag{7}$$

where T is number of observations, and $y_i, Hybrid_i$ are the actual and estimated values at point i, respectively.

8. Optimize weights using the BFGS algorithm [16] [17]: update the weights using a gradient descent method to obtain the optimal values. This is done by following these steps:
   - Initialize the weights w_1 and w_2 using random values.
   - Defining the problem: the mean square error (MSE) is calculated using the equation below [18],[19]:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}\left(D_{Acual_i} - D_{Estimated_i}\right)^2 \quad (8)$$

- Update weights using the equations:

$$w_1 = w_{1_0} + \alpha \frac{\partial MSE}{\partial w_1} \quad (9)$$

$$w_2 = w_{2_0} + \alpha \frac{\partial MSE}{\partial w_2} \quad (10)$$

  where α is the learning rate

- Repeat the steps to get the optimal weights.

9. Iterate the process until the lowest possible error is achieved.
10. Output results: display the final estimated values and compared performance with traditional methods.

3. Practical Application

To demonstrate the efficiency of the proposed method, a practical application was conducted on real data representing the radon concentration and the density of traces in some areas of Baghdad, measured in units $\mathrm{Bq/m^3}$ [20].

First step: Newton's method interpolation is used to calculate the estimated values $\mathrm{N(x, y)}$ using equation (1), (2) and (3). Table 1 represent the value of these estimates, while figures 1 and 2, respectively, plot these results and plot of the square errors at each point.

Table1. Results for Newton's interpolation method with errors

No.	Effect rate (x)	Exp.	Theor. Newton's interpolation method	Error	Square error
1	23.33	115.710000	115.709808	0.000192	3.6864×10^{-8}
2	9.66	47.930000	47.929726	0.000274	7.5076×10^{-8}
3	22.77	112.950000	112.951530	0.001530	2.3409×10^{-6}
4	26.11	129.480000	129.479751	0.000249	6.2001×10^{-8}
5	3.22	15.970000	15.968936	0.001064	1.1321×10^{-6}
6	5.44	26.990000	26.991603	0.001603	2.5696×10^{-6}
7	10.33	51.240000	51.241235	0.001235	1.5252×10^{-6}
8	18.22	90.360000	90.359770	0.000230	5.2900×10^{-8}
9	18	89.260000	89.258494	0.001506	2.2680×10^{-6}
10	36.11	179.070000	179.069555	0.000445	1.9802×10^{-7}
11	19.44	96.420000	96.419844	0.000156	2.4336×10^{-8}
12	16.66	82.650000	82.650276	0.000276	7.6176×10^{-8}
Sum				0.008760	7.6738×10^{-5}

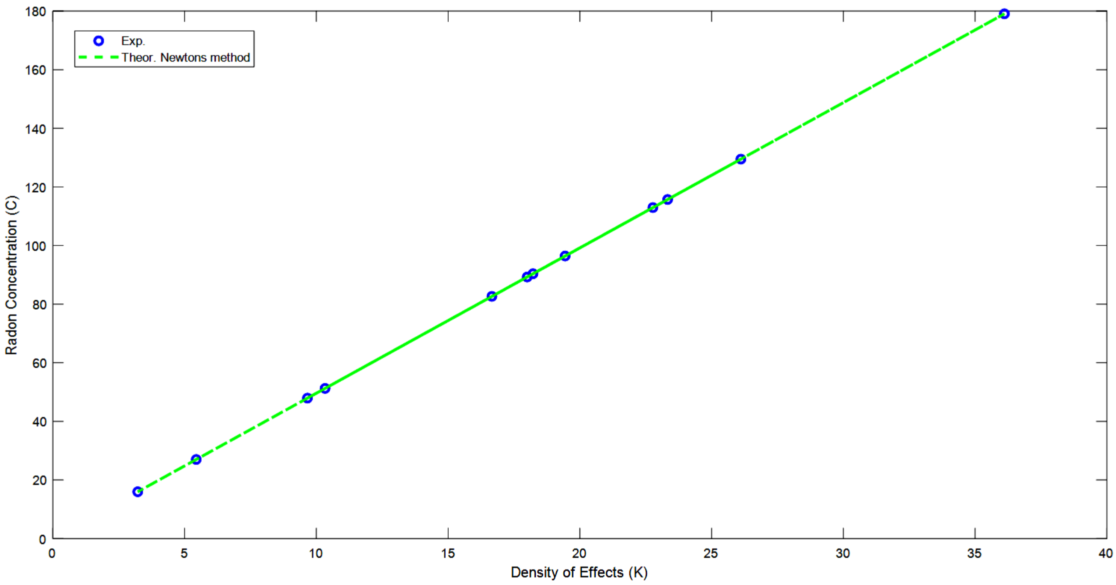

Fig. 1. results for newton's interpolation method

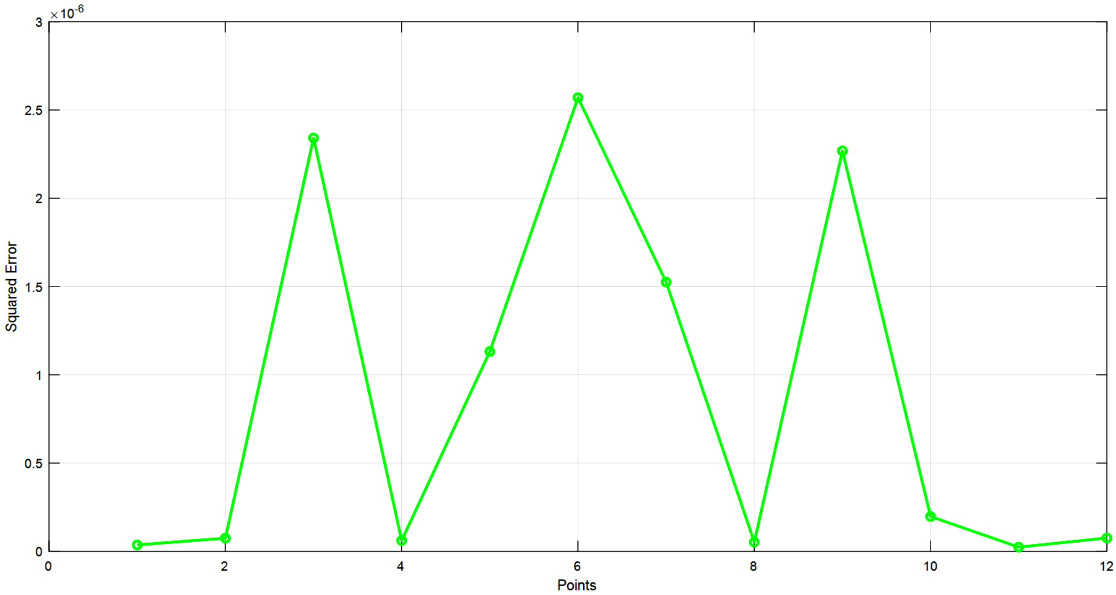

Fig. 2. squared errors - newton's interpolation method

From table 1 along with the absolute and squared errors at each point. It can be been that the total squared for Newton's interpolation method was 0.00876. From figure 1 the theoretical (estimated) values are plotted the experimental (actual) values at each data point. It can see the Newton's interpolation method provides estimated close to the actual values, but exhibits some small errors, this figure illustrates how Newton's interpolation method works in estimating values between known points. While in figure 2 the squared errors can be seen in Newton's interpolation method are relatively small, but they accumulate at the number of points increases, leading to an increase in the overall error. this figure demonstrates that newton's method suffers from some small errors accumulate as the number of points increases.

Second step: Lagrange method interpolation is used to calculate the estimated values $L(x, y)$ using equation (4) and (5). Table 2 represent the value of these estimates, while figures 3 and 4, respectively, plot these results and plot of the square errors at each point.

Table 2. Results for Newton's interpolation method with errors

No.	Effect rate (x)	Exp.	Theor. Lagrange method	Error	Square error
1	23.33	115.710000	115.710315	0.000315	9.9225×10^{-8}
2	9.66	47.930000	47.930227	0.000227	5.1529×10^{-8}
3	22.77	112.950000	112.950313	0.000313	9.7969×10^{-8}
4	26.11	129.480000	129.480326	0.000326	1.0628×10^{-7}
5	3.22	15.970000	15.970117	0.000117	1.3689×10^{-8}
6	5.44	26.990000	26.990169	0.000169	2.8561×10^{-8}
7	10.33	51.240000	51.240234	0.000234	5.4756×10^{-8}
8	18.22	90.360000	90.360290	0.000290	8.4100×10^{-8}
9	18	89.260000	89.260289	0.000289	8.3521×10^{-8}
10	36.11	179.070000	179.070359	0.000359	1.2888×10^{-7}
11	19.44	96.420000	96.420297	0.000297	8.8209×10^{-8}
12	16.66	82.650000	82.650281	0.000281	7.8961×10^{-8}
Sum				0.003216	1.0343×10^{-5}

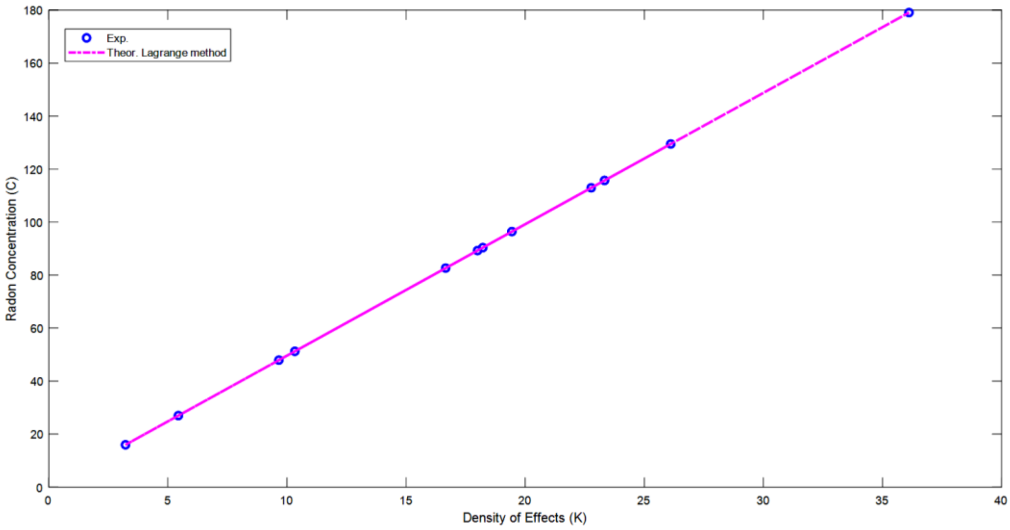

Fig. 3. results for Lagrange interpolation method

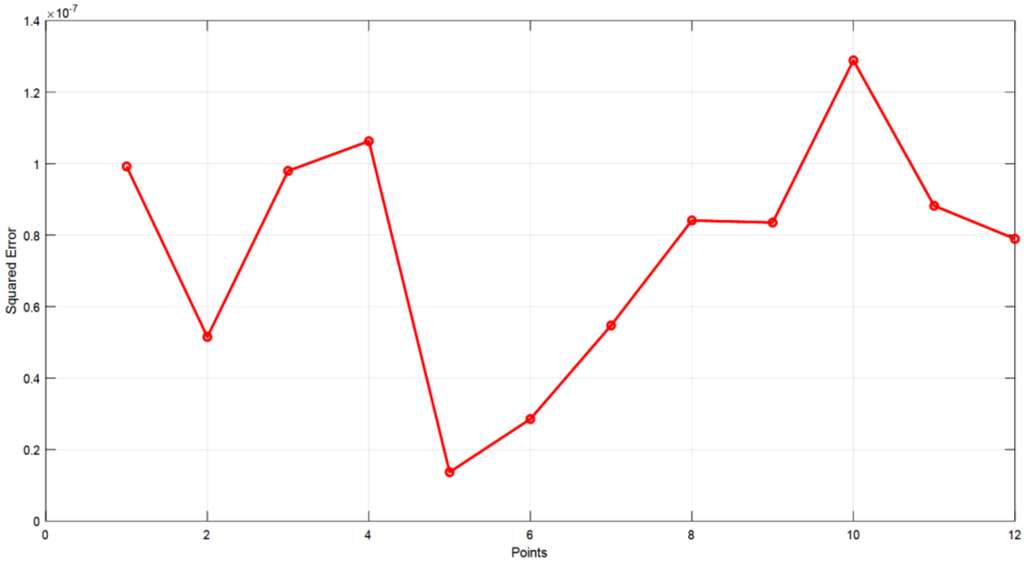

Fig. 4. Squared errors - Lagrange interpolation method

From table 2 along with the absolute and squared errors at each point. It can be been that the total squared for Lagrange method was 0.003216, indicating that this method is more accurate than Newton's method. From figure 3 the theoretical (estimated) values are plotted the experimental (actual) values at each data point. It can see the Lagrange method provides more accurate estimates than Newton's method, with smaller errors at most points, this figure demonstrates that the Lagrange method is more accurate than Newton's method at estimating values between known points. While in figure 4 the squared errors can be seen in Lagrange method are smaller than those of Newton's method, indicating that the Lagrange method is more accurate. However, some small errors remain that accumulate at the number of points increases. This figure demonstrates that Lagrange method is more accurate than Newton's method, but it still suffers from some small errors.

Third step: appropriate weights are given according to the estimation accuracy for each method at each point using BFGS algorithm, table 3 represents the estimated weights for each method and for all points.

Table3. Weights of the Newton's interpolation and Lagrange methods for each point

No.	Newton's interpolation method	Lagrange method	Sum
1	0.620514	0.379486	1
2	0.452999	0.547001	1
3	0.169978	0.830022	1
4	0.566874	0.433126	1
5	0.099681	0.900319	1
6	0.095996	0.904004	1
7	0.159507	0.840493	1
8	0.558084	0.441916	1
9	0.161383	0.838617	1
10	0.446621	0.553379	1
11	0.654828	0.345172	1
12	0.504677	0.495323	1

Table 3 represent the optimal weights for both the Newton's and Lagrange methods at each point using the BFGS algorithm. It can be seen that the weights change dynamically based on the accuracy of each point method at each point, contributing to the overall performance improvement of hybrid NL method. These results indicate that while the hybrid NL model is stable under small weight variations, large deviations from the optimized weights can lead to significant performance degradation. This underscores the importance of BFGS based optimization step in achieving accurate predictions.

Fourth step: the results of the hybrid NL method defined in equation 7 are calculating based on the weights of table 3 and the results of the estimated values from tables 1 and 2. Table 4 represent the value of hybrid NL method estimates, while figures 3 and 4, respectively, plot these results and plot of the square errors at each point of hybrid NL method.

Table4. Results for hybrid method with errors

No.	Effect rate (x)	Exp.	Theor. Lagrange method	Error	Square error
1	23.33	115.710000	115.710000	0.000000	0.000000
2	9.66	47.930000	47.930000	0.000000	0.000000
3	22.77	112.950000	112.950519	0.000519	2.6936×10^{-7}
4	26.11	129.480000	129.480000	0.000000	0.000000
5	3.22	15.970000	15.969999	0.000001	1×10^{-12}
6	5.44	26.990000	26.990307	0.000307	9.4249×10^{-8}

7	10.33	51.240000	51.240393	0.000393	1.5445×10^{-7}
8	18.22	90.360000	90.360000	0.000000	0.000000
9	18	89.260000	89.259999	0.000001	1×10^{-12}
10	36.11	179.070000	179.070000	0.000000	0.000000
11	19.44	96.420000	96.420000	0.000000	0.000000
12	16.66	82.650000	82.650279	0.000279	7.7841×10^{-8}
Sum				0.001501	2.2530×10^{-6}

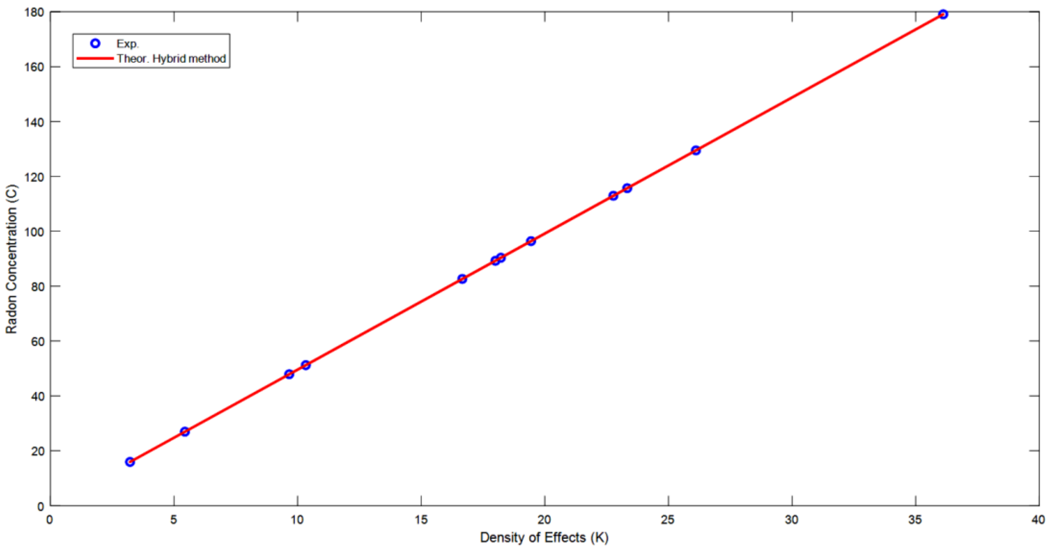

Fig. 5. results for hybrid NL method interpolation method

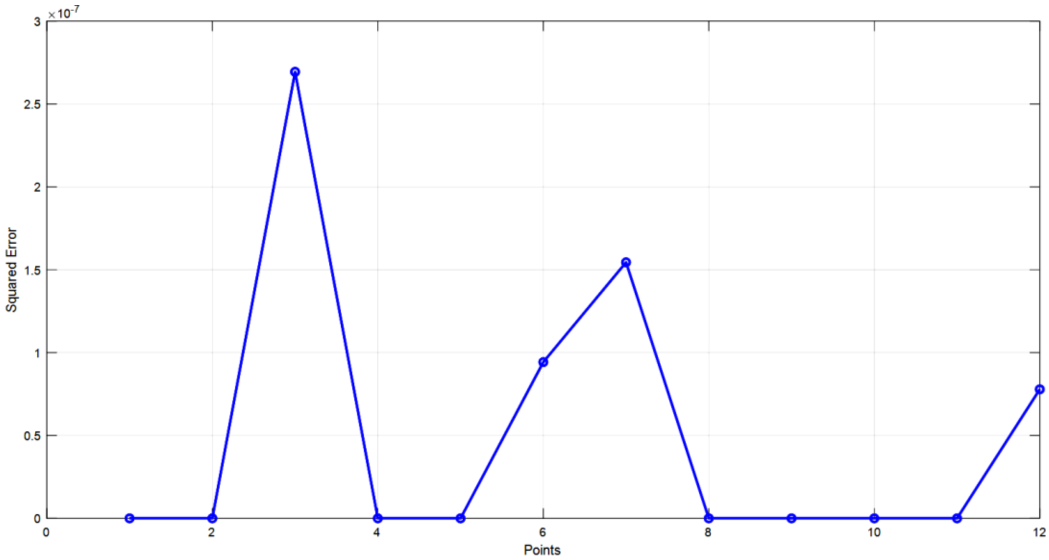

Fig. 6. squared Errors - hybrid NL method

From table 4 along with the absolute and squared errors at each point. It can be been that the total squared for Hybrid NL method was 0.001501, indicating that this method is more accurate than traditional methods. From figure 5 the theoretical (estimated) values are plotted the experimental (actual) values at each data point. It can see the Hybrid NL method provides more accurate estimates than traditional methods, with smaller errors at most points, this figure demonstrates that the Hybrid NL method is more accurate than traditional methods at estimating values between known points. While in figure 6 the squared errors can be seen that the squared errors of Hybrid NL are much smaller than those of Newton's method and Lagrange method, indicating that the Hybrid NL method is more accurate. This figure demonstrates that Hybrid NL method is more accurate than traditional methods, that the Hybrid NL is able to significantly reduce squared errors compared to traditional methods.

Fifth step, we will analyze the comparative performance of the three interpolation methods. Figures 7, 8, 9, and 10 represent the Performance of Each method in terms of absolute error, Relative Error Comparison, and squared errors. In additional, detailed results for each table represent in this study will be discussed to illustrate the effectiveness of the hybrid NL method in improving prediction accuracy compared to traditional methods.

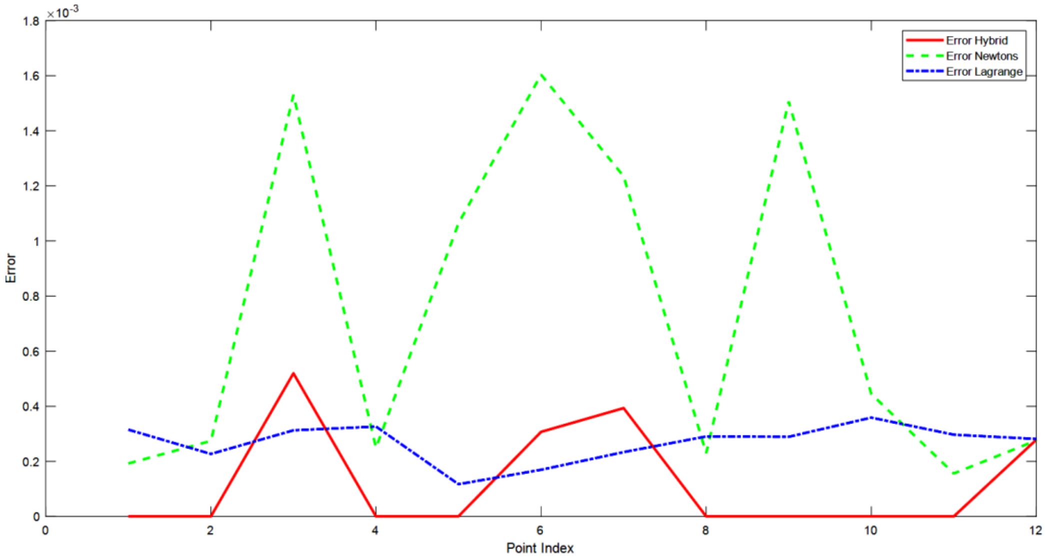

Fig. 7. errors for each method

From figure 7 it can see that the hybrid NL method exhibits superior performance with lower errors compared to Newton's method and Lagrange method. This indicates that the dynamic weight distribution using the BFGS algorithm has contributed to a significant error reduction.

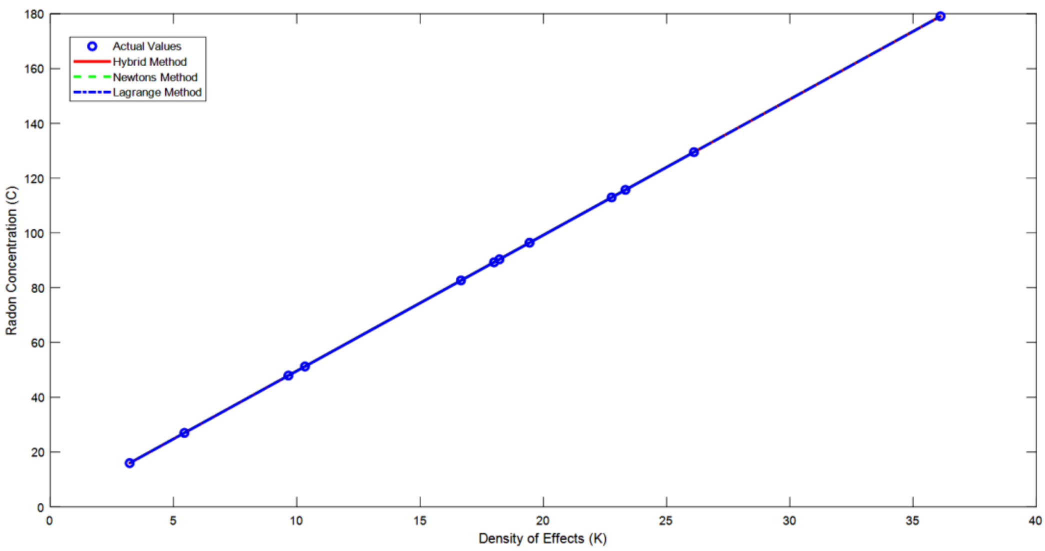

Fig. 8. Performance of each method

From figure 8 the hybrid method shows better performance, with significantly lower squared error, compared to the traditional methods. This confirms the effectiveness of the hybrid method in improving prediction accuracy.

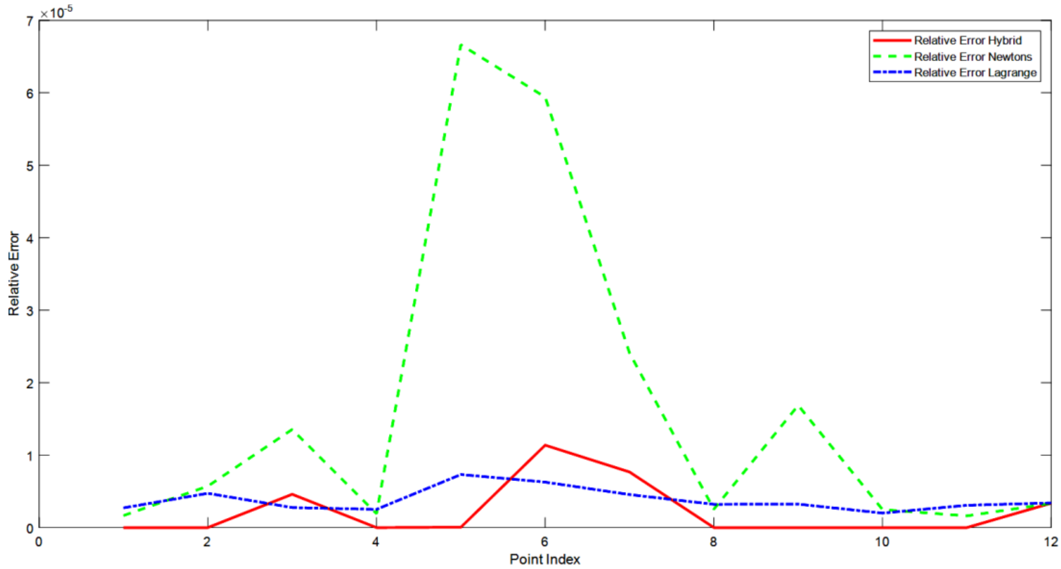

Fig. 9. relative Error comparison

From figure 9 the hybrid NL method exhibits lower relative errors than other methods, indicating that the hybrid NL method is more accurate at estimating values between known points.

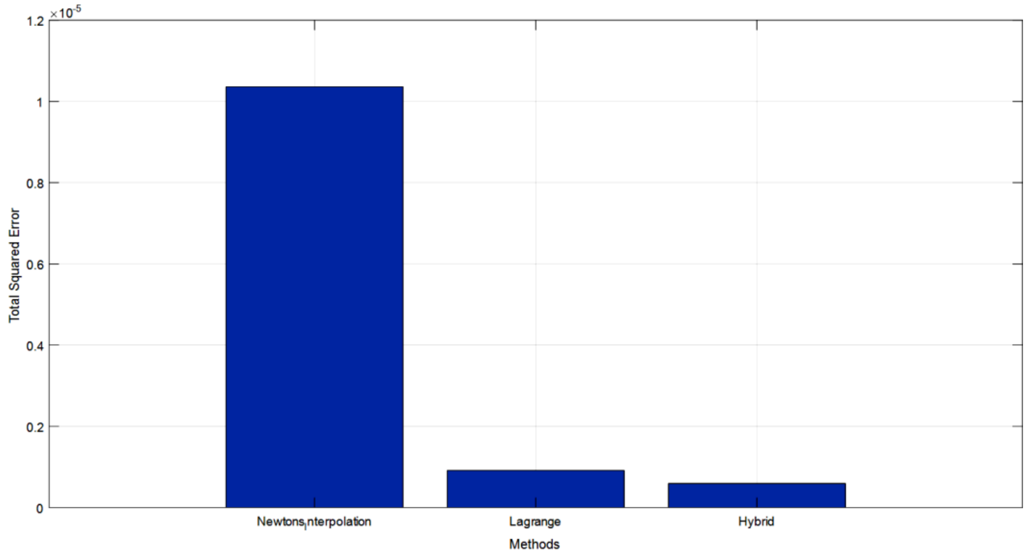

Fig. 10. performance of methods based on total squared errors

From figure 10 the hybrid NL method exhibits superior performance, with significantly lower total squared error compared to the traditional methods. This indicates that the hybrid NL method more effective in improving prediction accuracy.

4. Results and discussion

In this section, we will discuss the results obtained from the study. The hybrid NL method developed in this study demonstrated superior performance compared to traditional methods. This superior performance was achieved through dynamic weight distribution using BFGS algorithm, which improved the accuracy of estimations and significantly reduced errors. The comparison of the results between the three methods is as follows:

- Newton's method: this method showed a total error of 0.00876. indicating that it is less accurate than hybrid NL method. However, this method is more efficient when adding new points without having to recalculate all the terms.

- Lagrange method: this method showed a total error of 0.003216. indicating that it is more accurate than Newton's method but less accurate than hybrid NL method. This method is more accurate when dealing with specific data but suffer from increased computational complexity when dealing with large datasets.
- Hybrid NL method: this method showed a total error of 0.001501. indicating that it is more accurate than traditional methods. This superior performance was archived through dynamic weight distribution using BFGS algorithm, which allowed for significantly improved estimation accuracy and reduced errors.

Conclusions

This study developed a hybrid method that combines Newton's method and Lagrange method using BFGS algorithm for dynamic weight distribution. Theoretical results demonstrated that this hybrid method is capable of significantly improving the accuracy of estimations and reducing errors compared to traditional methods. In practice, the hybrid method was applied to real data representing the radon gas concentration in some areas of Baghdad. The results showed that the hybrid method was more accurate than traditional methods, demonstrating its effectiveness in improving the accuracy of predictions when dealing with real data. Based on the results obtained, the hybrid method can be recommended for use in other applications such as temporal data analysis and dynamic systems analysis. Furthermore, this method can be developed to improve the accuracy of predictions in other fields such as engineering and economics.

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